# DOCSIS 4.0 Profile Management Optimization – Moving Closer to Shannon Capacity

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## 1. Introduction

The recent introduction of Data over Cable Service Interface Specifications (DOCSIS® [1]) 4.0 technology has the potential to greatly increase the bandwidth and the resultant throughput of internet voice, video, and data services. This transition will provide a significant increase in the available spectrum, especially for upstream traffic (greater than a factor of 10).

Traditional operational practice has been to set cable modem upstream transmit levels to receive a constant receive power versus frequency at the node. While this practice has little consequence for prior versions of DOCSIS, the significant increase in available upstream spectrum may not achieve the full potential increase in capacity with this approach.

This paper explores the optimization of upstream capacity from an information theoretic viewpoint. A profile management application (PMA) is derived from the well-known Shannon channel capacity theorem. The maximization of the sum rate of all upstream channels using Shannon's formula employs a well-known optimization solution known as "water-filling". Novel methods tailoring this solution to the DOCSIS protocol are derived with examples of the increase in available capacity achievable in currently deployed cable systems.

It should be noted that both DOCSIS 4.0 full-duplex (FDX) and extended spectrum (ESD) DOCSIS specifications can benefit from this approach as the upstream spectrum using orthogonal frequency-division multiple access (OFDMA) transmission are the same. Also note that a similar power allocation optimization has been suggested in several 5G wireless radio orthogonal frequency-division multiplexing (OFDM) implementations (references in [2]).

## 2. Cable System Upstream Model

The model used in this paper for the PMA analysis for optimization of upstream capacity (bitrate) is shown in Figure 1.

The DOCSIS 4.0 cable modem can transmit across N parallel channels simultaneously in the 108 to 684 MHz band in either the Full Duplex DOCSIS (FDX) or Frequency Division Duplex (FDD) mode of operation, where N ranges from one to six upstream channels. Each upstream channel occupies a bandwidth of 96 MHz. A minimum of two active channels are required as the transmit power and corresponding receive level of each channel can be chosen differently for the optimization of capacity.

The transmitted signal power over the $i^{th}$ channel is denoted $s_i$, the received signal power over that channel is denoted $r_i$, and the path gain (magnitude transfer function) between the transmitting modem and the node receiver is denoted $g_i$. The path gain is mainly the loss from the modem through the tap port and across the taps in the span to the first amp in the upstream path. After reaching the first amp, unity gain in a properly configured plant will preserve the delivered signal level but add noise through the amplifier cascade. Noise power is added in each channel denoted $N_i$.

The path gains $g_i$ can be calculated by subtracting the modem reported transmit level after ranging from the measured modem receive level at the node port. For an FDX node with echo cancellation (EC) of the downstream echo from the upstream signal in the FDX band (108 to 684 MHz), the node post-EC residual level and signal-to-noise ratio (SNR) can be measured when no modems are granted transmission opportunities. The upstream receiver modulation error ratio (MER) using the minislot pilot tones can be

[1] CableLabs is the owner of the DOCSIS® trademark

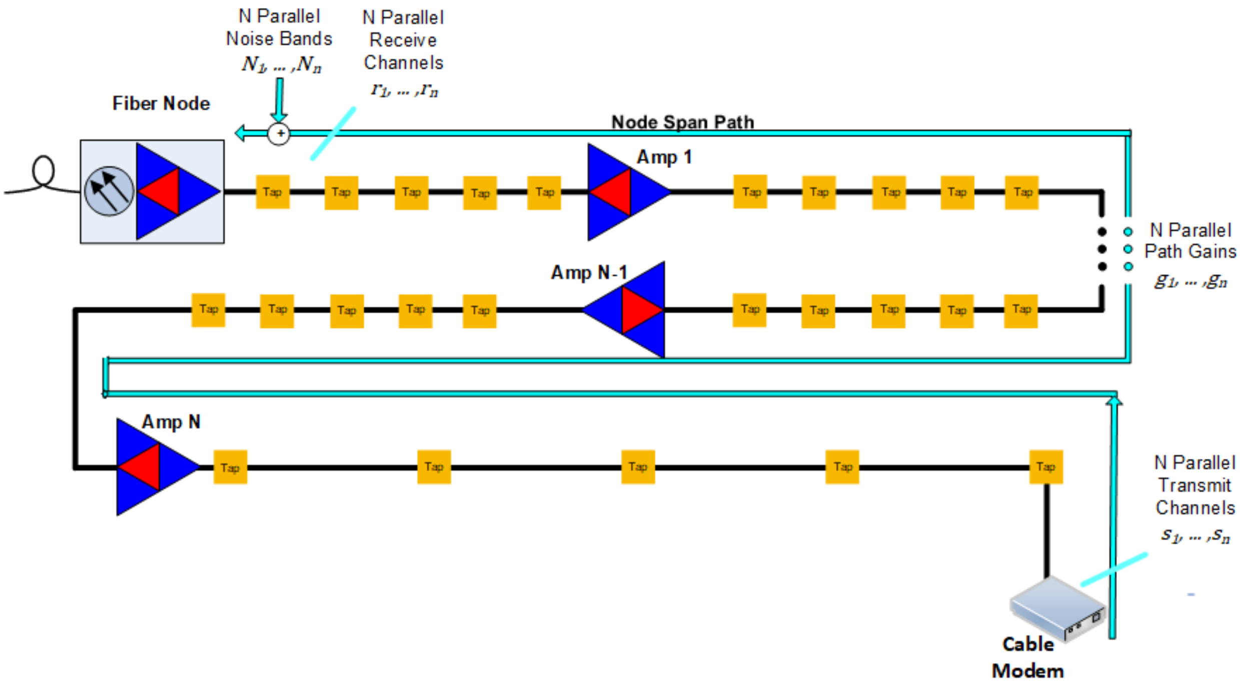



**Figure 1 – Cable System Upstream Transmission and Reception Model**

measured for a modem grant of a known channel transmit level. The input noise level $N_i$ to the node can then be calculated at the node. The received MER for each channel is available from the OFDMA demodulator which provides the received SNR for each upstream channel. These parameters will be used in the PMA algorithms to determine the optimum individual modem channel transmit levels to maximize capacity (i.e., bitrate) summed across the entire set of upstream OFDMA channels.

An example network previously analyzed in [1] will be used for the purpose of analysis of the techniques to be discussed here and is shown in Figure 2. A single leg branch is depicted with node, amp, tap, and drop design levels for a six-amplifier cascade. The cable modem that is transmitting upstream is attached to a 100-foot drop from the last tap after the last amplifier furthest from the node. The forthcoming analysis is done for FDX operation with node upstream receive levels of 8 and 12 dBmV/6.4 MHz. A variety of network designs exist, and the technique described in this paper is easily extensible to the myriad actual network configurations in production hybrid-fiber-coax (HFC) networks.

The signal-to-noise plus interference ratio (SNIR) thresholds for error-free reception are tabulated in Table 1 (denoted CNR). The modulation order and efficiency (bit-loading) for the quadrature amplitude modulation (QAM) levels are associated with these minimum CNR thresholds.

First the case of configuring this transmitting modem for equal receive levels across all six 96 MHz upstream channels at the input port of the node is considered. The modem upstream transmit level, node upstream receive level, and upstream bit-loading are shown for an upstream receive levels of 8 dBmV/6.4 MHz in Figure 3.

Table 2 tabulates the node port transmit and receive signal levels, echo interference levels, node signal to interference ratio (SIR), amp cascade SNR, and the combined node SNIR for this total modem transmit power. Note that the channel SNIR value in the bottom row of Table 2 is between the threshold for the 1024-QAM and 2048-QAM modulation order in Table 1. Hence lower threshold just below the 34 dB SNR is conservatively chosen with the bit-loading for 1024-QAM or 10 bits/symbol as shown in Figure 3.

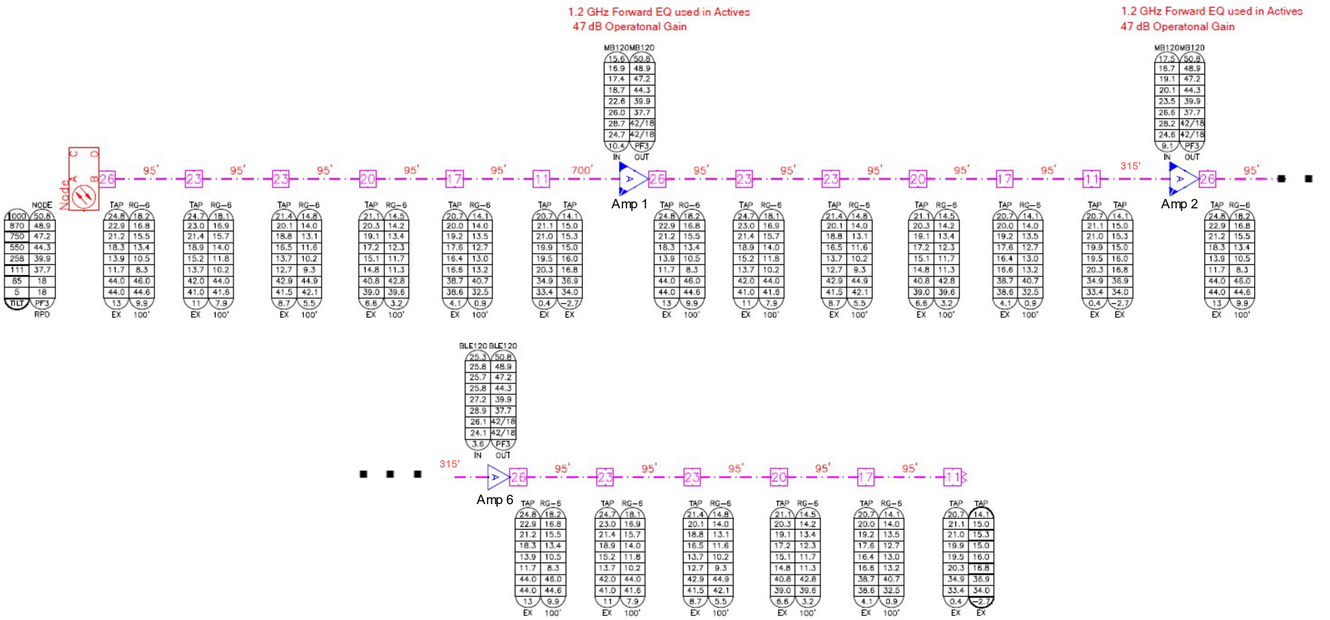
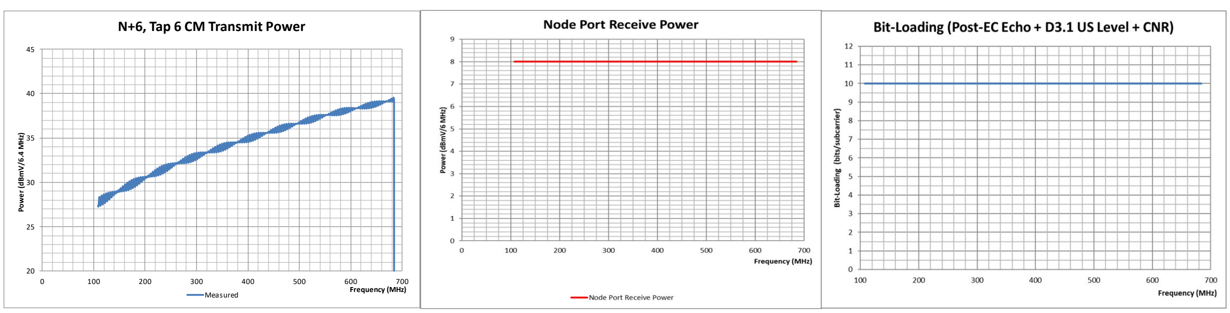



**Figure 2 – Single-Family Unit (SFU) Cable System for a Node + 6 Amplifier Cascade**




| Tap 6 CM Transmit Power | |
|---|---|
| Sub-Band Range (MHz) | CM Transmit Power (dBmV) |
| 108 - 300 | 45.6 |
| 300 - 492 | 49.7 |
| 492 - 684 | 52.9 |
| 108 - 684 | 55.1 |

| Average Bit-Loading Post-EC Echo + D3.1 US Rx + CNR 108 - 684 MHz | Upstream Capacity Post-EC Echo + D3.1 US Rx + CNIR 108 - 684 MHz |
|---|---|
| 10 bits/subc | 5,760 Mbps |

**Figure 3 – SFU Cable System Modem Transmit Level, Node Receive Level, and Upstream Bit-Loading for Upstream Receive Level of 8 dBmV/6.4 MHz and an Upstream Transmit Total Power of 55 dBmV**

## Table 1 – OFDMA Modulation Order vs. CNR Threshold for Error-Free Performance

| QAM Order | Modulation Efficiency (bits/subc | CNR Threshold (dB)* |
|---|---|---|
| 0 | 0.0 | -100.0 |
| QPSK | 2.0 | 8.5 |
| 8-QAM | 3.0 | 12.0 |
| 16-QAM | 4.0 | 15.0 |
| 32-QAM | 5.0 | 18.0 |
| 64-QAM | 6.0 | 20.5 |
| 128-QAM | 7.0 | 23.5 |
| 256-QAM | 8.0 | 26.5 |
| 512-QAM | 9.0 | 29.5 |
| 1024-QAM | 10.0 | 32.0 |
| 2048-QAM | 11.0 | 36.0 |
| 4096-QAM | 12.0 | 40.0 |

## Table 2 – SFU Cable System Node Transmit Level, Node Receive Level, and Upstream Receive SNR for an Upstream Transmit Total Power of 55 dBmV

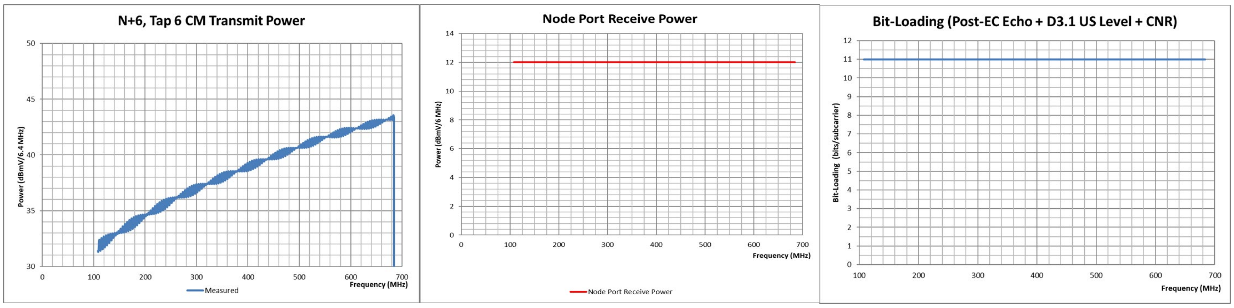
| Node +6 | Upstream Channels | | | | | |
|---|---|---|---|---|---|---|
| Upstream Frequency Band: | 108 - 204 MHz | 204 - 300 MHz | 300 - 396 MHz | 396 - 492 MHz | 492 - 588 MHz | 588 - 684 MHz |
| Node Port Transmit Power (dBmV): | 44.4 | 45.8 | 47.3 | 48.7 | 50.1 | 51.5 |
| Node Port Echo Power (dBmV): | 19.8 | 20.7 | 21.9 | 23.1 | 24.3 | 25.6 |
| Node DC Tap Leakage Power (dBmV): | 6.7 | 8.1 | 9.6 | 11.0 | 12.4 | 13.8 |
| Node DC Tap Echo + Leakage Power (dBmV): | 20.0 | 20.9 | 22.1 | 23.4 | 24.6 | 25.9 |
| Node Port Receive Power (dBmV/6.4 MHz): | 8.0 | 8.0 | 8.0 | 8.0 | 8.0 | 8.0 |
| Node RPD Post-EC Average Echo Residual (dBmV/6.4 MHz): | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 |
| Node Port Post-EC Average S/[Echo + Leakage] Ratio (dB): | 43.5 | 43.5 | 43.5 | 43.5 | 43.5 | 43.5 |
| Node Port Average CM + Amps Background SNR (dB): | 34.7 | 34.7 | 34.7 | 34.7 | 34.7 | 34.7 |
| **Node RPD Post-EC SNIR (dB):** | 34.1 | 34.1 | 34.1 | 34.1 | 34.1 | 34.2 |

## Table 3 – SFU Cable System Node Transmit Level, Node Receive Level, and Upstream Receive SNR for an Upstream Transmit Total Power of 59 dBmV

| Node +6 | Upstream Channels | | | | | |
|---|---|---|---|---|---|---|
| Upstream Frequency Band: | 108 - 204 MHz | 204 - 300 MHz | 300 - 396 MHz | 396 - 492 MHz | 492 - 588 MHz | 588 - 684 MHz |
| Node Port Transmit Power (dBmV): | 44.4 | 45.8 | 47.3 | 48.7 | 50.1 | 51.5 |
| Node Port Echo Power (dBmV): | 19.8 | 20.7 | 21.9 | 23.1 | 24.3 | 25.6 |
| Node DC Tap Leakage Power (dBmV): | 6.7 | 8.1 | 9.6 | 11.0 | 12.4 | 13.8 |
| Node DC Tap Echo + Leakage Power (dBmV): | 20.0 | 20.9 | 22.1 | 23.4 | 24.6 | 25.9 |
| Node Port Receive Power (dBmV/6.4 MHz): | 12.0 | 12.0 | 12.0 | 12.0 | 12.0 | 12.0 |
| Node RPD Post-EC Average Echo Residual (dBmV/6.4 MHz): | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 |
| Node Port Post-EC Average S/[Echo + Leakage] Ratio (dB): | 47.5 | 47.5 | 47.5 | 47.5 | 47.5 | 47.5 |
| Node Port Average CM + Amps Background SNR (dB): | 37.8 | 37.8 | 37.8 | 37.8 | 37.8 | 37.8 |
| **Node RPD Post-EC SNIR (dB):** | 37.3 | 37.3 | 37.3 | 37.3 | 37.4 | 37.4 |

The modem upstream transmit level, node upstream receive level, and upstream bit-loading are shown for an upstream receive levels of 12 dBmV/6.4 MHz in Figure 4.

Table 3 tabulates the node port transmit and receive signal levels, echo interference levels, node SIR, amp cascade SNR, and the combined node SNIR for this total modem transmit power. Note that the channel SNR value in the bottom row of Table 3 is between the threshold for the 2048-QAM and 4096-QAM modulation order in Table 1. Hence the lower threshold just below the 37 dB SNR is conservatively chosen with the bit-loading for 2048-QAM or 11 bits/symbol as shown in Figure 4.




| Tap 6 CM Transmit Power | |
|---|---|
| Sub-Band Range (MHz) | CM Transmit Power (dBmV) |
| 108 - 300 | 49.6 |
| 300 - 492 | 53.7 |
| 492 - 684 | 56.9 |
| 108 - 684 | 59.1 |

| Average Bit-Loading Post-EC Echo + D3.1 US Rx + CNR 108 - 684 MHz | Upstream Capacity Post-EC Echo + D3.1 US Rx + CNIR 108 - 684 MHz |
|---|---|
| 11 bits/subc | 6,336 Mbps |

**Figure 4 – SFU Cable System Modem Transmit Level, Node Receive Level, and Upstream Bit-Loading for Upstream Receive Level of 12 dBmV/6.4 MHz and an Upstream Transmit Total Power of 59 dBmV**

## 3. Capacity and Maximum Sum Rate with Conventional Water-Filling

In this section, the channel capacity of the cable system model of Figure 1 is analyzed. The maximum throughput rate of the i$^{th}$ channel is given by the well-known Shannon channel capacity [3] $C_i$ in channel bandwidth $w_i$ as:

$$C_i = w_i \, log_2\,(1 + SNR_i) = \, w_i \, log_2 \left(1 + \frac{g_i s_i}{N_i}\right)$$

*(1)*

where $C = \sum_{i=1}^{n} C_i$; $C_i = w_i\big(log_2(1 + SNR_i)\big)$; $g_i$ = path loss; $s_i$ = transmitted signal power; $N_i$ = received noise power, and $SNR_i = \frac{g_i s_i}{N_i}$ for i channels.

The total throughput is found by the sum of rates over all n channels as:

$$\sum_{i=1}^{n} w_i \, log_2 \left(1 + \frac{g_i s_i}{N_i}\right)$$

*(2)*

The available transmit power P is constrained by the sum of all channel powers $s_i$ as:

$$\sum_{i=1}^{n} s_i = P$$

*(3)*

The optimal power loading per channel is found by solving the following sum rate maximization:

$$maximize \sum_{i=1}^{n} w_i \, log_2 \left(1 + \frac{g_i s_i}{N_i}\right) \; over \; s_i$$
$$such \; that \sum_{i=1}^{n} s_i = P$$
$$where \; s_i \geq 0 \;\; \forall \, i$$

*(4)*

The solution to this sum rate maximization is found by the Lagrange multiplier method described in Appendix 1 and the Sum Rate Maximization (4) is solved for the transmit powers $s_i$ in Appendix 2. This solution [3] is widely known as “water-filling” for reasons described below.

The Lagrangian $\mathcal{L}$ for the maximum sum rate objective with constraints in (4) derived in Appendix 2 is:

$$\mathcal{L}(\boldsymbol{s}, \boldsymbol{\lambda}, \lambda_0) = -\sum_{i=1}^{n} w_i \, log_2 \left(1 + \frac{g_i s_i}{N_i}\right) + \lambda_0 \left(\sum_{i=1}^{n} s_i - P\right) - \sum_{i=1}^{n} \lambda_i s_i$$

*(5)*

The solution is found by setting the gradient of the Lagrangian with respect to the power variables $s_i$ equal to zero, solving for the transmit power levels $s_i$, and applying the Karush-Kuhn-Tucker (KKT) conditions of complementary slackness to remove the slack variables $\{\lambda_1, \dots, \lambda_n\}$**.** The result is:

$$s_i = \left[\frac{1}{\lambda_0} - \frac{N_i}{g_i}\right]^+ = \max\left(\frac{1}{\lambda_0} - \frac{N_i}{g_i}, 0\right)$$

*(6)*

where [ ]$^+$ disallows negative power values. Let $\alpha = \frac{1}{\lambda_0}$, so that

$$s_i = \left[\alpha - \frac{N_i}{g_i}\right]^+$$

*(7)*

A common analogy compares $\alpha$ to a “water-filling” power level whereby all $s_i$ can be found as the difference between this water-filling level $\alpha$ and the inverse of the channel gain divided by the channel noise level $\frac{N_i}{g_i}$.

The analogy would be described as power being represented by water being poured into a tank with different depths in the bottom representing the inverse channel gain divided by noise level. The water level rises above all or most of the different bottom depths as shown in Figure 5.

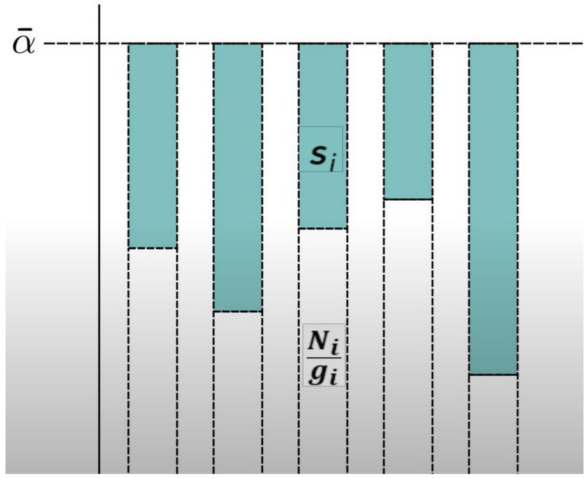



**Figure 5 – Water-filling level α filling power represented as water in each channel i to a depth of $N_i/g_i$ from the bottom of the tank.**

Note that the water-filling level α is implicitly specified and must be found by iteratively checking if the sum of the calculated transmit power levels in (7) is equal to the maximum total power constraint P in equation (4). This is an implicit solution as the water-filling level $\alpha$ can be found subject to the maximum power constraint $\sum_{i=1}^{n} s_i = P$ using an iterative procedure such as a bisection search for the value of $\alpha$ that meets this constraint and is denoted as the Conventional Water-Filling (CWF) Algorithm. Some possible bounds $\alpha_{Low}$ and $\alpha_{High}$ for the start of a bisection search for $\alpha$ are

Initialize:

$$\alpha_{Low} = min\left(\frac{N_i}{g_i}\right), i = 1, \dots, n \ \ (lowest\ active\ channel)$$
$$\alpha_{High} = \frac{P + \sum_{i=1}^{n} \frac{N_i}{g_i}}{n} \qquad (all\ channels\ active) \tag{8}$$

Iterative search:

1. $\alpha_{Next} = \left(\alpha_{Low} + \alpha_{High}\right)/2$
2. Calculate $s_i$ (7) with $\alpha_{Next}$
3. Calculate $\sum_{i=1}^{n} s_i = P'$
4. If $P' > P$ then $\alpha_{High} = \alpha_{Next}$, else $\alpha_{Low} = \alpha_{Next}$
5. If $|P' - P| < \varepsilon$ then STOP and output $\{s_i\}_{i=1}^{n}$, else go to 1

The initial $\alpha_{Low}$ and $\alpha_{High}$ bisection water level starting values and the final iteration water level typical results are shown in Figure 6. In theory an infinite number of bisection calculation will converge on the exact water level, but a tolerable error for $\varepsilon \to 0$ can limit the number of iterations.

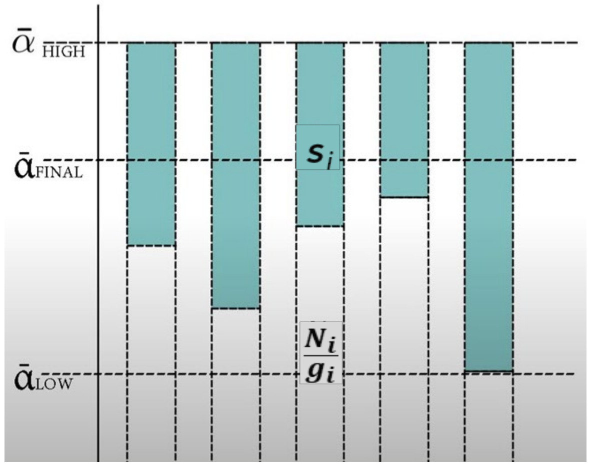



**Figure 6 – Water-filling initial levels $\alpha_{Low}$ and $\alpha_{High}$ and the final water level $\alpha_{Final}$ after bisection iterative search closely meeting the total sum limit of all active channel powers.**

## 4. Capacity and Maximum Sum Rate with Geometric Water-Filling

A novel approach to solve (4) has been proposed in [2]. The proposed Geometric Water-Filling (GWF) approach eliminates the procedure to solve the non-linear system for the water level and provides explicit solutions. Figure 7 (a)-(d) illustrates the GWF algorithm [2].

Suppose there are 4 steps/stairs (n = 4) with unit width inside a water tank. For the conventional approach, the dashed horizontal line, which is the water level α, needs to be determined first and then the power allocated for each stair (water volume above the stair) is solved.

Let $d_i$ denote the "step depth" of the i[th] stair which is the height of the i[th] step to the bottom of the tank, and is given by

$$d_i = \frac{1}{a_i w_i} \; where \; a_i = \frac{g_i}{N_i}, i = 1, \dots, n \tag{9}$$

The sequence $a_i$ is sorted as monotonically decreasing, then the step depth of the stairs indexed as $\{1, \dots, n\}$ is monotonically increasing. Define $\delta_{ij}$ as the "step depth difference" of the i[th] and the j[th] stairs expressed as

$$\delta_{ij} = d_i - d_j = \frac{1}{a_i w_i} - \frac{1}{a_j w_j}, i \geq j \; and \; 1 \leq i, j \leq n \tag{10}$$

Instead of trying to determine the water level α, which is a real non-negative number, determine the water level step, which is an integer number from 1 to n, denoted by $k^*$ as the highest step under water. Based on the result of $k^*$, the solutions for power allocation are determined explicitly without the knowledge of the water level α.

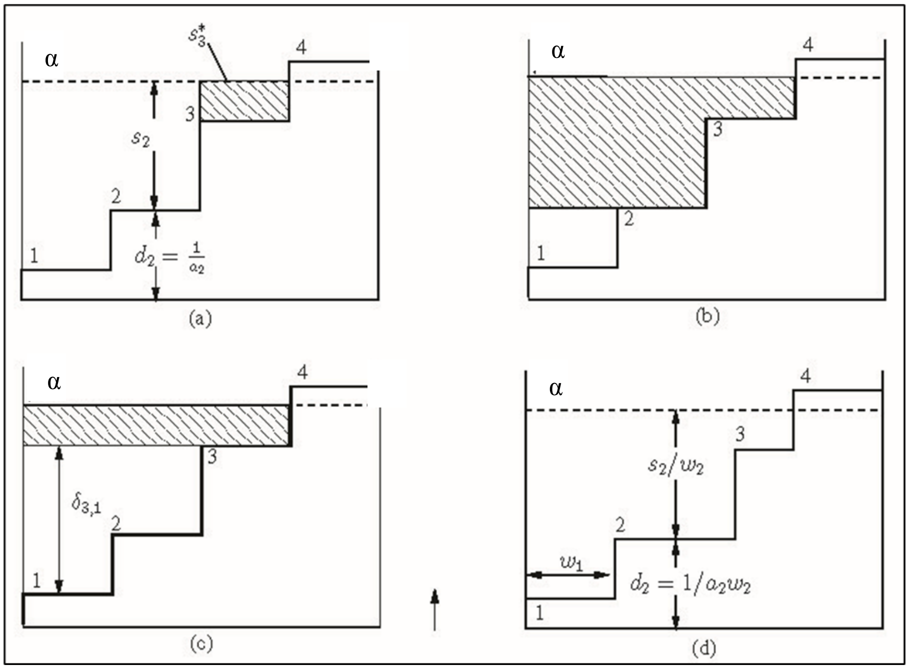



**Figure 7 – Illustration for the Geometric Water-Filling (GWF) algorithm.**

**(a) Illustration of water level step k∗ = 3, allocated power for the third step s3∗, and step/stair depth di = 1/ai. (b) Illustration of P2(k) (shadowed area, representing the total water/power above step k) when k = 2. (c) Illustration of P2(k) when k = 3. (d) Illustration of the weighted case.**

Let $P_2(k)$ denote the water volume above step k or zero, whichever is greater. The value of $P_2(k)$ can be solved by subtracting the volume of the water under step k from the total power P, as

$$P_2(k) = P - \left[\sum_{i=1}^{k-1} (d_k - d_i) w_i\right]^+ , k = 1, \dots, n \tag{11}$$

Due to the definition of $P_2(k)$ being the power (water volume) above step $k$, it cannot be a negative number (i.e., negative power). Therefore the use of $[\ ]^+$ in (11) assigns 0 to $P_2(k)$ if the result inside the bracket is negative. The corresponding geometric meaning is that the $k^{th}$ level is above water.

The explicit solution to (4) is:

$$s_i = \begin{cases} \left[\frac{s_{k^*}}{w_{k^*}} + (d_{k^*} - d_i)\right] w_i, 1 \le i \le k^* \\ 0, \qquad\qquad\qquad\qquad k^* < i \le n \end{cases} \tag{12}$$

where the water level step $k^*$ is given as

$$k^* = max\{k | P_2(k) > 0\}, \qquad 1 \le k \le n \tag{13}$$

and the power level for this step is

$$s_{k^*} = \frac{w_{k^*}}{\sum_{i=1}^{k^*} w_i} P_2(k^*) \tag{14}$$

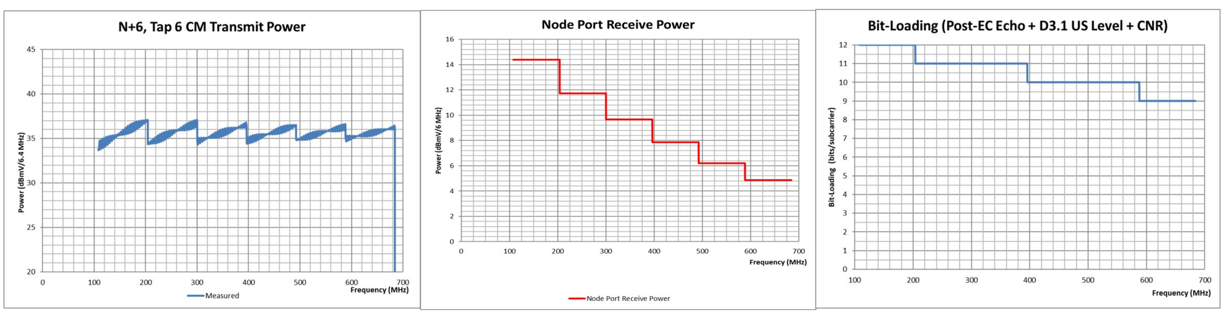
## 5. Example of Geometric Water-Filling

The GWF algorithm described in the previous section was analyzed in the cable network of Figure 2. The cable modem that is transmitting upstream is attached to a 100-foot RG-6 drop cable from the last tap after the last amplifier furthest from the node.

The transmitting modem receive levels at the node are calculated across all six 96 MHz upstream channels using the above described GWF algorithm.

The modem upstream transmit level, node upstream receive level, and upstream bit-loading are shown for comparable modem total transmit powers of Figure 3 in Figure 8.

The total modem transmit power is 55.2 dBmV which is the same as the 8 dBmV/6.4 MHz receive level case of Figure 3. The node receiver bit-loading is 10.5 bits/symbol with a capacity (bitrate) of 6049 Mbps. The GWF algorithm results of Figure 8 increases modem capacity over the uniform node receive level of Figure 3 and yields an increase of 0.5 bit/symbol average across the six upstream channel with a resultant 6049/5760 = 5 percent increase in bitrate.




| Tap 6 CM Transmit Power | |
|---|---|
| Sub-Band Range (MHz) | CM Transmit Power (dBmV) |
| 108 - 300 | 50.5 |
| 300 - 492 | 50.4 |
| 492 - 684 | 50.4 |
| 108 - 684 | 55.2 |

| Average Bit-Loading Post-EC Echo + D3.1 US Rx + CNR 108 - 684 MHz | Upstream Capacity Post-EC Echo + D3.1 US Rx + CNIR 108 - 684 MHz |
|---|---|
| 10.5 bits/subc | 6,049 Mbps |

**Figure 8 – SFU Cable System Modem Transmit Level, Node Receive Level, and Upstream Bit-Loading for an Upstream Transmit Total Power of 55 dBmV for CWF**

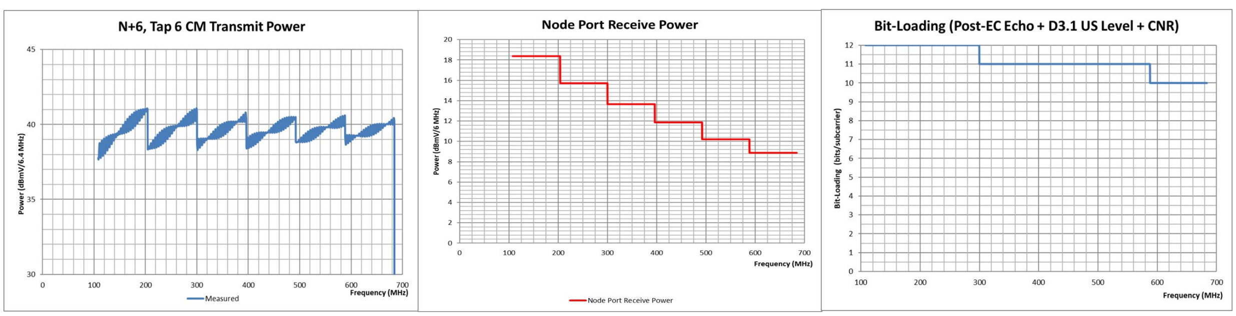
Table 4 tabulates the node port transmit and receive signal levels, echo interference levels, node self-interference SIR, amp cascade SNR, and the combined node SNIR for this total modem transmit power. Note that several channel SNR values in the bottom row of Table 4 are above the thresholds for the modulation orders in Table 1. Also, several SNR values are nearly equal to the closest SNR threshold.

Table 4 – SFU Cable System Node Transmit Level, Node Receive Level, and Upstream Receive SNR for an Upstream Transmit Total Power of 55 dBmV for GWF

| Node + 6 | Upstream Channels | | | | | |
|---|---|---|---|---|---|---|
| Upstream Frequency Band: | 108 - 204 MHz | 204 - 300 MHz | 300 - 396 MHz | 396 - 492 MHz | 492 - 588 MHz | 588 - 684 MHz |
| Node Port Transmit Power (dBmV): | 44.4 | 45.8 | 47.3 | 48.7 | 50.1 | 51.5 |
| Node Port Echo Power (dBmV): | 19.8 | 20.7 | 21.9 | 23.1 | 24.3 | 25.6 |
| Node DC Tap Leakage Power (dBmV): | 6.7 | 8.1 | 9.6 | 11.0 | 12.4 | 13.8 |
| Node DC Tap Echo + Leakage Power (dBmV): | 20.0 | 20.9 | 22.1 | 23.4 | 24.6 | 25.9 |
| Node Port Receive Power (dBmV/6.4 MHz): | 14.3 | 11.6 | 9.6 | 7.8 | 6.1 | 4.8 |
| Node RPD Post-EC Average Echo Residual (dBmV/6.4 MHz): | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 |
| Node Port Post-EC Average S/[Echo + Leakage] Ratio (dB): | 49.8 | 47.1 | 45.1 | 43.3 | 41.6 | 40.3 |
| Node Port Average CM + Amps Background SNR (dB): | 41.3 | 38.7 | 36.6 | 34.8 | 33.2 | 31.8 |
| **Node RPD Post-EC SNIR (dB):** | 40.7 | 38.1 | 36.0 | 34.2 | 32.6 | 31.3 |

The transmitting modem receive levels at the node for a total modem transmit power of 55.2 dBmV are calculated across all six 96 MHz upstream channels using the above described GWF algorithm. The modem upstream transmit level, node upstream receive level, and upstream bit-loading are shown for comparable modem total transmit powers of Figure 4 in Figure 9.

The total modem transmit power is 59.2 dBmV which is the same as the 12 dBmV/6.4 MHz receive level case of Figure 4. The node receiver bit-loading is 11.2 bits/symbol with a capacity (bitrate) of 6432 Mbps. The GWF algorithm results of Figure 9 increases modem capacity over the uniform node receive level of Figure 4 and yields an increase of 0.2 bit/symbol average across the six upstream channel with a resultant 6432/6336 = 1.5 percent increase in bitrate.




| Tap 6 CM Transmit Power | |
|---|---|
| Sub-Band Range (MHz) | CM Transmit Power (dBmV) |
| 108 - 300 | 54.5 |
| 300 - 492 | 54.4 |
| 492 - 684 | 54.4 |
| 108 - 684 | 59.2 |

| Average Bit-Loading Post-EC Echo + D3.1 US Rx + CNR 108 - 684 MHz | Upstream Capacity Post-EC Echo + D3.1 US Rx + CNIR 108 - 684 MHz |
|---|---|
| 11.2 bits/subc | 6,432 Mbps |

Figure 9 – SFU Cable System Modem Transmit Level, Node Receive Level, and Upstream Bit-Loading for an Upstream Transmit Total Power of 59 dBmV for GWF

The transmitting modem receive levels at the node for a total modem transmit power of 59.2 dBmV are calculated across all six 96 MHz upstream channels using the above described GWF algorithm. Table 5 tabulates the node port transmit and receive signal levels, echo interference levels, node self-interference SIR, amp cascade SNR, and the combined node SNIR for this total modem transmit power.

Again, note that many channel SNR values in the bottom row of Table 5 are above the thresholds for the modulation orders in Table 1. Also, several SNR values are nearly equal to the closest SNR threshold.

**Table 5 – SFU Cable System Node Transmit Level, Node Receive Level, and Upstream Receive SNR for an Upstream Transmit Total Power of 59 dBmV for CWF**

| **Node + 6** | Upstream Channels | | | | | |
|---|---|---|---|---|---|---|
| Upstream Frequency Band: | 108 - 204 MHz | 204 - 300 MHz | 300 - 396 MHz | 396 - 492 MHz | 492 - 588 MHz | 588 - 684 MHz |
| Node Port Transmit Power (dBmV): | 44.4 | 45.8 | 47.3 | 48.7 | 50.1 | 51.5 |
| Node Port Echo Power (dBmV): | 19.8 | 20.7 | 21.9 | 23.1 | 24.3 | 25.6 |
| Node DC Tap Leakage Power (dBmV): | 6.7 | 8.1 | 9.6 | 11.0 | 12.4 | 13.8 |
| Node DC Tap Echo + Leakage Power (dBmV): | 20.0 | 20.9 | 22.1 | 23.4 | 24.6 | 25.9 |
| Node Port Receive Power (dBmV/6.4 MHz): | 18.3 | 15.6 | 13.6 | 11.8 | 10.1 | 8.8 |
| Node RPD Post-EC Average Echo Residual (dBmV/6.4 MHz): | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 |
| Node Port Post-EC Average S/[Echo + Leakage] Ratio (dB): | 53.8 | 51.1 | 49.1 | 47.3 | 45.6 | 44.3 |
| Node Port Average CM + Amps Background SNR (dB): | 44.8 | 42.1 | 40.1 | 38.3 | 36.6 | 35.3 |
| **Node RPD Post-EC SNIR (dB):** | 44.3 | 41.6 | 39.6 | 37.8 | 36.1 | 34.8 |

## 6. Geometric Water-Filling with Individual Peak Power Constraints

The weighted water-filling with individual peak power constraints problem is stated as follows. Given P > 0, as the total power or volume of the water, the allocated power and the path gain for the $i^{th}$ channel are given as $s_i$ and $a_i$ respectively, $i = 1, \ldots, n$ where $n$ is the total number of transmit channels. The weights $w_i > 0\ \forall i$, and without loss of generality, the array elements of $\left\{\frac{g_i}{N_i} w_i\right\}_{i=1}^{n}$ are positive and monotonically decreasing. The sum rate optimization with peak power per channel constraints becomes:

$$max_{\{s_i\}_{i=1}^{n}} \sum_{i=1}^{n} w_i\, log_2\left(1 + \frac{g_i s_i}{N_i}\right)$$

$$subject\ to\ \ 0 \le s_i \le P_i, \qquad \forall\ i$$

$$\sum_{i=1}^{n} s_i \le P \qquad (15)$$

Comparing the problem (15) with (4), the constraint of $0 \le s_i$ is extended to $0 \le s_i \le P_i$, i.e., additional individual peak power constraints, and $\sum_{i=1}^{n} s_i = P$ equality to $\sum_{i=1}^{n} s_i \le P$ inequality. The problem (15) is thus referred to as (weighted) water-filling with sum and individual peak power constraints (WFPP).

The previous section 4 provides an explicit solution using geometric view approach. Interestingly, the proposed GWF can be applied to the WFPP problem with some modifications. The following presents an algorithm which is a modification of the above discussed GWF, and it is termed as the geometric water-filling with sum and individual peak power constraints (GWFPP).

The previous expression (11) can be extended into the new expression:

$$P_2(i_k) = \left[P - \sum_{t=1}^{|E|-1} (d_{i_k} - d_{i_t}) w_{i_t}\right]^+, \qquad for\ k = 1, \ldots, |E|$$

where E is a subsequence of the sequence $\{1, 2, \ldots, n\}$, |E| is the cardinality of the set E. E can be expressed as $\{i_1, i_2, \ldots, i_{|E|}\}$. If E is taken as the sequence $\{1, 2, \ldots, n\}$, then the extended expression is regressed into the original expression (11). Similarly, some corresponding changes in (13)-(15) are also done (i.e., the subscripts of sequence are replaced with those of the subsequence). These extended expressions are still labelled as (12) - (14) in the following statement of Algorithm GWFPP.

Algorithm GWFPP:

Initialize:

$vectors\ \{d_i\}, \{w_i\}, \{P_i\}\ for\ i = 1, 2, \ldots, n$

$the\ set\ E = \{1, 2, \ldots, n\}$

total transmit power limit = $P$

Iterative search:

1. Use (12) - (14) to compute $\{s_i\}$.
2. Define the set $\Lambda = \{i | s_i > P_i, i \in E\}$.
3. If $\Lambda$ is the empty set, output $\{s_i\}_{i=1}^{n}$ and exit, else $s_i = P_i, i \in \Lambda$.
4. Update $E$ with $E \setminus \Lambda$ and $P$ with $P - \sum_{i \in \Lambda} P_i$ and return to step 1.

Algorithm GWFPP is a dynamic power distribution process. The state of this process is the difference between the individual peak power sequence and the current power distribution sequence obtained by the Algorithm GWF. The control of this process is to use (12) - (14) of the Algorithm GWF based on the state mentioned above. Thus, a new state for the next iteration appears. Therefore, an optimal dynamic power distribution process, the GWFPP, with the state feedback is formed. Since the finite set E is getting smaller and smaller until the set $\Lambda$ is empty, Algorithm GWFPP carries out at most $n$ loops (the number of channels) to compute the optimal solution.

## 7. Geometric Water-Filling with Individual Peak Power and Receive SNR Constraints

As previously noted in Section 5, several channel SNR values in the bottom row of Table 4 and Table 5 are above the thresholds for the modulation orders in Table 1. Imposing individual channel peak power constraints using the GWFPP algorithm on such channels, a modem transmit peak power limit can be imposed that corresponds to the node receive SNR threshold. This redistributes the excess power in such peak power limited channels to other channels that are close to but under an SNR threshold. This may increase the aggregate bit-loading by increasing the SNR in those underpowered channels.

As described in Section 2, the path gains $g_i$, the node post-EC residual level and SNR, the upstream receiver MER, the input noise level $N_i$ to the node, the received MER for each channel, and the OFDMA demodulator received SNR for each upstream channel can be determined. These parameters will be used in the PMA algorithms to determine the optimum individual modem channel transmit levels to maximize capacity (i.e., bitrate) summed across the entire set of upstream OFDMA channels.

These parameters provide inputs to the GWF algorithm to calculate the optimum transmit power levels constrained only by the sum of the individual channel transmit power levels. The procedure for a node plus amplifier cascade is as follows:

1. Set the maximum total power constraint. The same total power used in a flat receive level across all upstream channels could be used.
2. Set the modem target receive level and transmit a test burst from the modem being measured with a calibrated (ranged) transmit power level.
3. Measure the receive level at the node port and calculate the path gain as the difference of the node receive level minus the modem transmit level.

4. Measure the upstream receiver MER at the node demodulator.
5. Calculate the noise input level as the modem transmit level plus the path gain minus the upstream receiver MER.
6. Calculate the constrained optimum modem transmit powers per channel using the GWF algorithm of Section 4.

Next, the modem transmit power limit for each channel can be determined as each channel receive level adjusted to the nearest modulation order SNR threshold plus the path gain. The set of adjusted node receive power levels plus the path gain then become the input individual peak power constraints in the GWFPP algorithm. Perform the following GWFPP algorithm iterative procedure:

0. Set the maximum total power constraint to the absolute upper limit (for calculation purposes only, since the sum of the power per channel constraints will limit the actual total transmit power next).
1. Measure the resulting upstream receive SNR per channel and set the SNR thresholds to the closest value of the thresholds in Table 1.
2. Compute the individual peak power modem transmit levels for each channel as the upstream received SNR plus the input noise level minus the path gain.
3. Compute and set the modem upstream receive levels at the node port as the peak power modem transmit levels plus the path gain.
4. Recalculate/remeasure the resulting channel SNR values.
5. Repeat steps 1 to 4 until there is no additional change in SNR for all channels (within a small dB difference) and stop (SNR convergence achieved).

Each successive iterative calculation changes the received SNR. The algorithm converges as the magnitude of the SNR changes from the previous iteration diminishes.

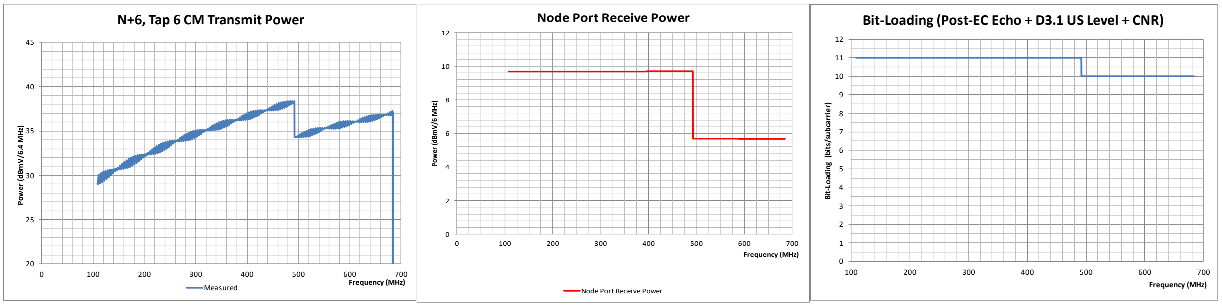
# 8. Example of Geometric Water-Filling with Individual Peak Power Constraints and Receive SNR Constraints

The GWFPP algorithm described in Section 7 was analyzed in the cable network of Figure 2. The cable modem that is transmitting upstream is attached to a 100-foot drop from the last tap after the last amplifier furthest from the node.

The transmitting modem receive levels at the node are calculated across all six 96 MHz upstream channels using the above described GWFPP algorithm. The modem upstream transmit level, node upstream receive level, and upstream bit-loading are shown for comparable modem total transmit powers of Figure 3 in Figure 10.

The total modem transmit power is 54.9 dBmV which essentially is the same as the 8 dBmV/6.4 MHz receive level case of Figure 3 at 55.1 dBmV. The node receiver bit-loading is 10.7 bits/symbol with a capacity (bitrate) of 6144 Mbps. The GWFPP algorithm results of Figure 10 increases modem capacity over the uniform node receive level of Figure 3 and yields an increase of 0.7 bit/symbol average across the six upstream channel with a resultant 6144 /5760 = 6.7 percent increase in bitrate.




| Tap 6 CM Transmit Power | |
|---|---|
| Sub-Band Range (MHz) | CM Transmit Power (dBmV) |
| 108 - 300 | 47.3 |
| 300 - 492 | 51.4 |
| 492 - 684 | 50.6 |
| 108 - 684 | 54.9 |

| Average Bit-Loading Post-EC Echo + D3.1 US Rx + CNR 108 - 684 MHz | Upstream Capacity Post-EC Echo + D3.1 US Rx + CNIR 108 - 684 MHz |
|---|---|
| 10.7 bits/subc | 6,144 Mbps |

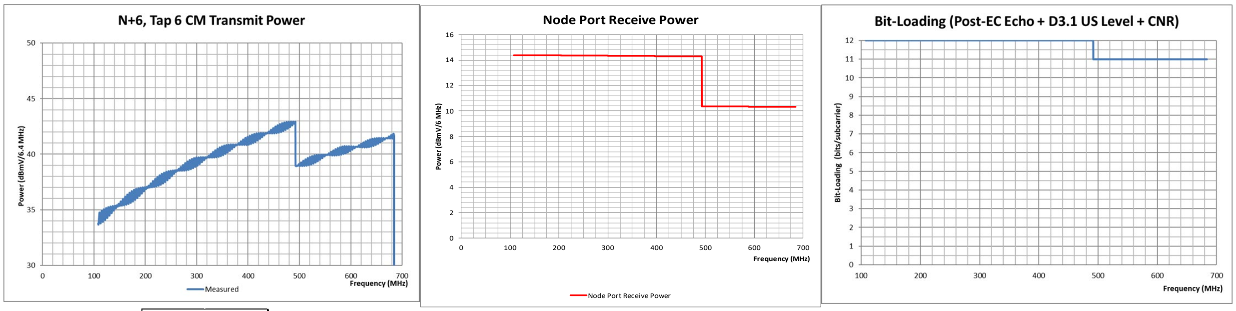
**Figure 10 – SFU Cable System Modem Transmit Level, Node Receive Level, and Upstream Bit-Loading for an Upstream Transmit Total Power of 55 dBmV for CWFPP**

Table 6 tabulates the node port transmit and receive signal levels, echo interference levels, node self-interference SIR, amp cascade SNR, and the combined node SNIR for this total modem transmit power. Note that all channel SNR values in the bottom row of Table 6 are now at the thresholds for the modulation orders in Table 1. Bit-loading has been redistributed with a net gain over the GWF algorithm from 5 percent to 6.7 percent over the Fixed Receive Level Channels case.

The transmitting modem receive levels at the node are calculated across all six 96 MHz upstream channels using the above described GWFPP algorithm. The modem upstream transmit level, node upstream receive level, and upstream bit-loading are shown for comparable modem total transmit powers of Figure 4 in Figure 11.

**Table 6 – SFU Cable System Node Transmit Level, Node Receive Level, and Upstream Receive SNR for an Upstream Transmit Total Power of 55 dBmV for GWFPP**

| **Node + 6** | Upstream Channels | | | | | |
|---|---|---|---|---|---|---|
| Upstream Frequency Band: | 108 - 204 MHz | 204 - 300 MHz | 300 - 396 MHz | 396 - 492 MHz | 492 - 588 MHz | 588 - 684 MHz |
| Node Port Transmit Power (dBmV): | 44.4 | 45.8 | 47.3 | 48.7 | 50.1 | 51.5 |
| Node Port Echo Power (dBmV): | 19.8 | 20.7 | 21.9 | 23.1 | 24.3 | 25.6 |
| Node DC Tap Leakage Power (dBmV): | 6.7 | 8.1 | 9.6 | 11.0 | 12.4 | 13.8 |
| Node DC Tap Echo + Leakage Power (dBmV): | 20.0 | 20.9 | 22.1 | 23.4 | 24.6 | 25.9 |
| Node Port Receive Power (dBmV/6.4 MHz): | 9.7 | 9.7 | 9.7 | 9.7 | 5.7 | 5.7 |
| Node RPD Post-EC Average Echo Residual (dBmV/6.4 MHz): | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 |
| Node Port Post-EC Average S/[Echo + Leakage] Ratio (dB): | 45.2 | 45.2 | 45.2 | 45.2 | 41.2 | 41.2 |
| Node Port Average CM + Amps Background SNR (dB): | 36.7 | 36.7 | 36.7 | 36.6 | 32.7 | 32.7 |
| **Node RPD Post-EC SNIR (dB):** | 36.1 | 36.1 | 36.1 | 36.1 | 32.1 | 32.1 |




| Tap 6 CM Transmit Power | |
|---|---|
| Sub-Band Range (MHz) | CM Transmit Power (dBmV) |
| 108 - 300 | 52.0 |
| 300 - 492 | 56.0 |
| 492 - 684 | 55.2 |
| 108 - 684 | 59.5 |

| Average Bit-Loading Post-EC Echo + D3.1 US Rx + CNR 108 - 684 MHz | Upstream Capacity Post-EC Echo + D3.1 US Rx + CNIR 108 - 684 MHz |
|---|---|
| 11.7 bits/subc | 6,720 Mbps |

**Figure 11 – SFU Cable System Modem Transmit Level, Node Receive Level, and Upstream Bit-Loading for an Upstream Transmit Total Power of 59.5 dBmV for CWFPP**

The total modem transmit power is 59.5 dBmV which is about the same as the 12 dBmV/6.4 MHz receive level case of Figure 4 at 59.1 dBmV. The node receiver bit-loading is 11.7 bits/symbol with a capacity (bitrate) of 6720 Mbps. The GWFPP algorithm results of Figure 11 increases modem capacity over the uniform node receive level of Figure 4 and yields an increase of 0.7 bit/symbol average across the six upstream channel with a resultant 6720/6336 = 6.1 percent increase in bitrate.

The transmitting modem receive levels at the node for a total modem transmit power of 59.2 dBmV are calculated across all six 96 MHz upstream channels using the above described GWF algorithm. Table 5 tabulates the node port transmit and receive signal levels, echo interference levels, node self-interference SIR, amp cascade SNR, and the combined node SNIR for this total modem transmit power.

**Table 7 – SFU Cable System Node Transmit Level, Node Receive Level, and Upstream Receive SNR for an Upstream Transmit Total Power of 59 dBmV for GWFPP**

| Node + 6 | Upstream Channels | | | | | |
|---|---|---|---|---|---|---|
| Upstream Frequency Band: | 108 - 204 MHz | 204 - 300 MHz | 300 - 396 MHz | 396 - 492 MHz | 492 - 588 MHz | 588 - 684 MHz |
| Node Port Transmit Power (dBmV): | 44.4 | 45.8 | 47.3 | 48.7 | 50.1 | 51.5 |
| Node Port Echo Power (dBmV): | 19.8 | 20.7 | 21.9 | 23.1 | 24.3 | 25.6 |
| Node DC Tap Leakage Power (dBmV): | 6.7 | 8.1 | 9.6 | 11.0 | 12.4 | 13.8 |
| Node DC Tap Echo + Leakage Power (dBmV): | 20.0 | 20.9 | 22.1 | 23.4 | 24.6 | 25.9 |
| Node Port Receive Power (dBmV/6.4 MHz): | 14.3 | 14.3 | 14.3 | 14.3 | 10.3 | 10.3 |
| Node RPD Post-EC Average Echo Residual (dBmV/6.4 MHz): | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 | -43.5 |
| Node Port Post-EC Average S/[Echo + Leakage] Ratio (dB): | 49.8 | 49.8 | 49.8 | 49.8 | 45.8 | 45.8 |
| Node Port Average CM + Amps Background SNR (dB): | 40.6 | 40.6 | 40.6 | 40.6 | 36.6 | 36.6 |
| **Node RPD Post-EC SNIR (dB):** | 40.1 | 40.1 | 40.1 | 40.1 | 36.1 | 36.1 |

The results of the use of water-filling with a total modem transmit power constraint including individual peak power constraints is summarized in Table 8. Note that Geometric Water-Filling with Peak Power and Receive SNR Constraints provides at least a 6 percent increase in capacity over Fixed Receive Level Channels of the same modem total transmit power.

Table 7 tabulates the node port transmit and receive signal levels, echo interference levels, node self-interference SIR, amp cascade SNR, and the combined node SNIR for this total modem transmit power. Note that all channel SNR values in the bottom row of Table 7 are now at the thresholds for the modulation orders in Table 1. Bit-loading has been redistributed with a net gain over the GWF algorithm from 1.5 percent to 6 percent over the Fixed Receive Level Channels case.

The results of the use of conventional Fixed Receive Level Channels and both water-filling algorithms with a total modem transmit power constraint including individual peak power constraints is summarized in Table 8. The optimization using Water-Filling with Peak Power and Receive SNR Constraints has been demonstrated to be superior.

**Table 8 – Profile Management Application Capacity Increase with Water-Filling Methods**

| **Fixed Receive Level Channels (8 and 12 dBmV/6.4 MHz) (55 and 59 dBmV Total Power)** | | | **Geometric Water-Filling** | | | **Geometric Water-Filling with Peak Power and Receive SNR Constraints** | | |
|---|---|---|---|---|---|---|---|---|
| Transmit Power (dBmV) | Average Bit-Loading (bits/symbol) | Upstream Capacity (Mbps) | Transmit Power (dBmV) | Average Bit-Loading (bits/symbol) | Upstream Capacity (Mbps) | Transmit Power (dBmV) | Average Bit-Loading (bits/symbol) | Upstream Capacity (Mbps) |
| 55.1 | 10 | 5760 | 55.2 | 10.5 | 6049 (+5%) | 54.9 | 10.7 | 6144 (+6.7%) |
| 59.1 | 11 | 6336 | 59.2 | 11.2 | 6432 (+1.5%) | 59.5 | 11.7 | 6720 (+6%) |

## 9. Conclusion

Water-filling is well-known. But a new advanced DOCSIS-based water-filling solution that can improve capacity within the existing DOCSIS protocols and cable modem operation is a new and unique aspect. The optimization of upstream capacity/bitrate using the sum rate maximization of the Shannon Channel Capacity bound across multiple upstream channels was analyzed. A novel previously published method to graphically solve this problem explicitly was used instead of the conventional implicit iterative approach (bisection search).

The results show that the solution to this sum rate maximization yields a continuous function of SNR. However, the OFDM modulation used in DOCSIS 4.0 is a discontinuous function of capacity versus SNR.

A further optimization was shown to increase the realized OFDMA channel capacity without increasing the total transmit power constraint using individual peak power limits on each channel. The peak power values are derived from the SNR thresholds for the OFDM modulation orders. Capacity improvements were shown to be at least 6 percent for a typical network example within the same or lower modem total transmit power. More bits/s/Hz increases system capacity within the same power and bandwidth. More capacity means more Mbps/$ spent on plant upgrades.

This new optimization provides a novel Profile Management Application methodology for DOCSIS 4.0 FDX and FDD modes of operation. Operational efficiency is improved. As a result, higher service tiers can be better supported or more headroom for existing service tiers is provided. More headroom increases reliability for providing more consistent service to customers.

## 10. Appendix 1 – The Lagrangian Function

A constrained optimization problem seeks to maximize (or minimize) an objective multivariable function $f(x, y, \ldots)$ subject to the constraint that another multivariable function equals a constant $g(x, y, \ldots) = c$. Joseph Louis Lagrange studied these constrained optimization problems and found a clever way to express all such conditions into a single equation [Lagrangian].

For the case of only one constraint and only two variables in the objective function, these conditions can be expressed by looking for constants that satisfy the following conditions:

The constraint:

$$g(x_0, y_0) = c$$

The tangency condition:

$$\nabla f(x_0, y_0) = \lambda_0 \nabla \mathrm{g}(x_0, y_0)$$

This can be broken into its components as follows:

$$\frac{\delta}{\delta x} f(x_0, y_0) = \lambda_0 \frac{\delta}{\delta x} \mathrm{g}(x_0, y_0)$$

$$\frac{\delta}{\delta y} f(x_0, y_0) = \lambda_0 \frac{\delta}{\delta y} \mathrm{g}(x_0, y_0)$$

Lagrange proposed a special new function which takes in all the same input variables as $f$ and g along with $\lambda$, thought of now as a variable rather than a constant.

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

Notice, the partial derivative of $\mathcal{L}$ with respect to $\lambda$ is $-(g(x, y) - c)$.

$$\frac{\delta}{\delta \lambda} \mathcal{L}(x, y, \lambda) = 0 - (g(x, y) - c)$$

So this can translate the condition $g(x, y) = c$ as

$$\frac{\delta}{\delta \lambda} \mathcal{L}(x, y, \lambda) = -g(x, y) + c = 0, \text{ or}$$

$$g(x, y) = c$$

Setting the other partial derivatives of $\mathcal{L}$ with respect to $x$ and $y$ equal to 0 yields

$$\frac{\delta}{\delta x} \mathcal{L}(x, y, \lambda) = \frac{\delta}{\delta x} f(x, y) - \lambda \frac{\delta}{\delta x} g(x, y) = 0, \text{ or}$$

$$\frac{\delta}{\delta x} f(x, y) = \lambda \frac{\delta}{\delta x} g(x, y)$$

and

$$\frac{\delta}{\delta y} \mathcal{L}(x, y, \lambda) = \frac{\delta}{\delta y} f(x, y) - \lambda \frac{\delta}{\delta y} g(x, y) = 0, \text{ or}$$

$$\frac{\delta}{\delta y} f(x, y) = \lambda \frac{\delta}{\delta y} g(x, y).$$

Together, these conditions are the same as saying:

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

Therefore, the three conditions needed to solve to find $x, y$ and $\lambda$ come down to the various partial derivatives of $\mathcal{L}$ being equal to 0. This can be written in extremely compact form by setting the gradient of $\mathcal{L}$ equal to the zero vector:

$$\nabla \mathcal{L} = \mathbf{0}$$

This function $\mathcal{L}$ is called the "Lagrangian", and the new variable $\lambda$ is called a "Lagrange multiplier". The Lagrange multiplier $\lambda$ is the rate of change of the quantity being optimized as a function of the constraint parameter.

The Lagrange multiplier technique finds the maximum or minimum of a multivariable function $f(x, y, \ldots)$ when there is some constraint on the input variable values. The Lagrange multiplier essentially enforces the constraint.

This technique only applies to constraints that look something like $g(x, y, \ldots) = c$, where $g$ is another multivariable function with the same input space as $f$, and $c$ is some constant.

For example, if the input space is two-dimensional, the graph of $f(x, y)$ with the line representing $g(x, y) = c$ projected onto it might look something like this:

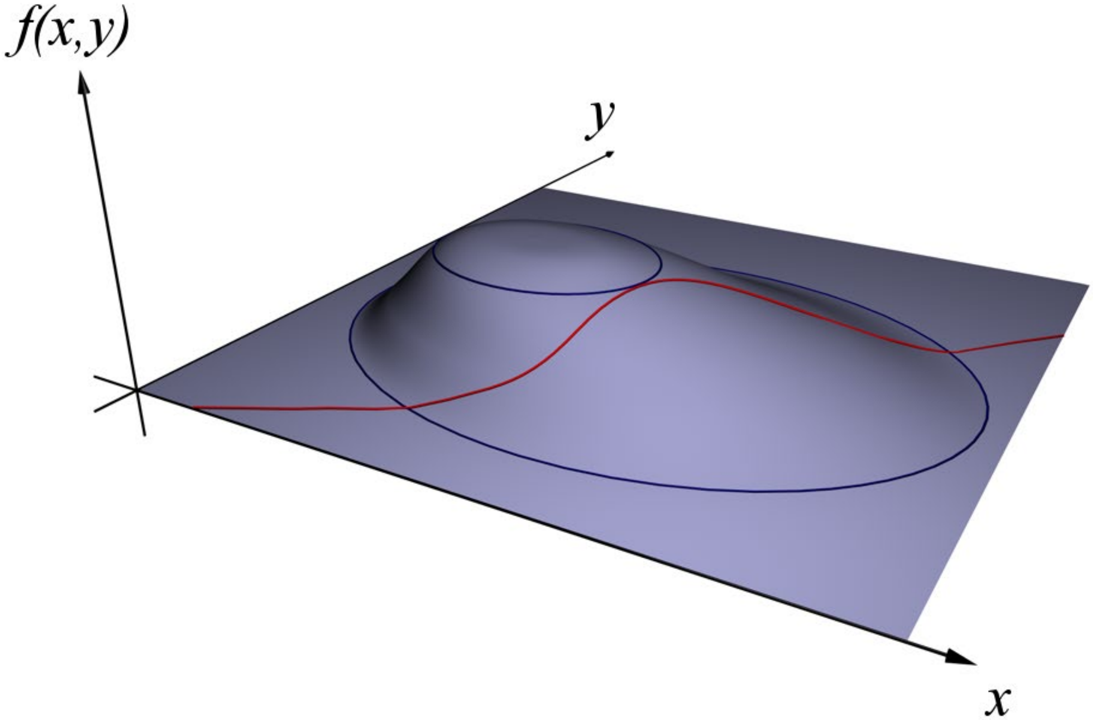



Image credit: By Nexcis (Own work) [Public domain], via Wikimedia Commons

**Figure 12 – Constrained optimization solution where the gradient of the Lagrangian equals the zero vector (3-D view)**

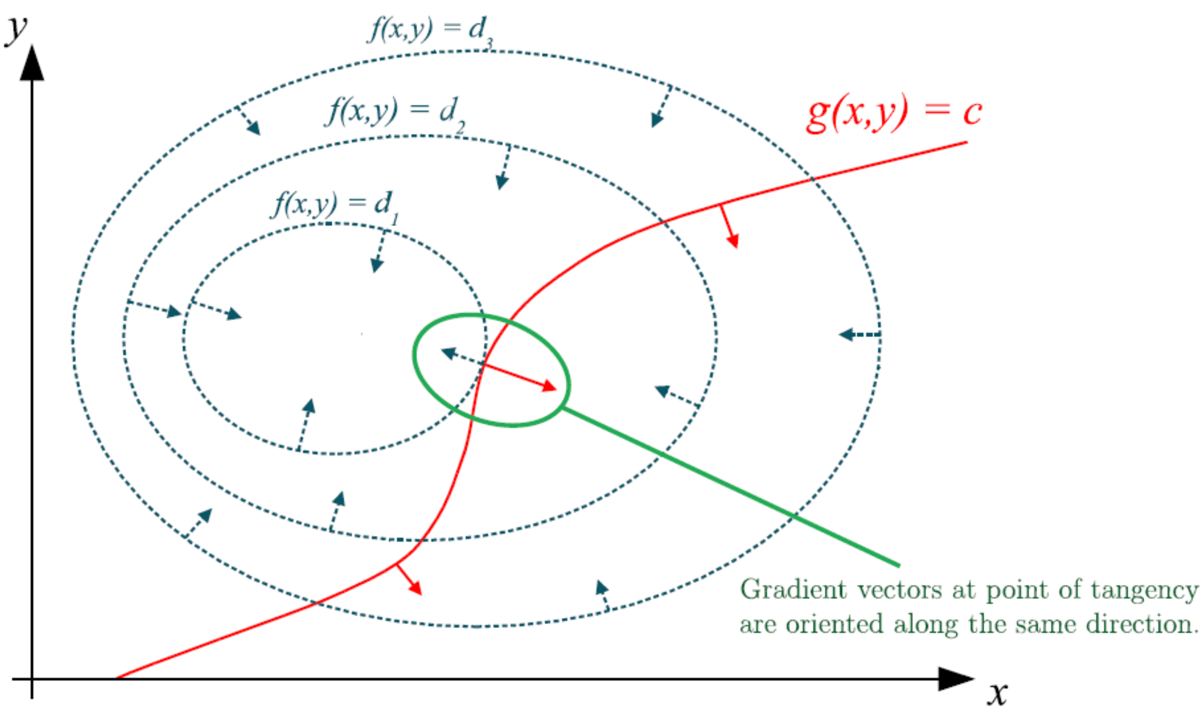



Image credit: By Nexcis (Own work) [Public domain], via Wikimedia Commons

**Figure 13 – Constrained optimization solution where the gradient of the Lagrangian equals the zero vector (2-D contour view)**

The goal is to find the highest point on that red line.

- The core idea is to look for points where the contour lines of $f$ and $g$ are tangent to each other.
- This is the same as finding points where the gradient vectors of $f$ and $g$ are parallel to each other.
- The entire process can be boiled down into setting the gradient of the Lagrangian equal to the zero vector.

To maximize (or minimize) a multivariable function $f(x, y, \dots)$ subject to the constraint that another multivariable function equals a constant $g(x, y, \dots) = c$:

- Introduce a new variable $\lambda$ (the Lagrange multiplier), and define a new function $\mathcal{L}$ (the Lagrangian function) as follows:

  $$\mathcal{L}(x, y, \dots, \lambda) = f(x, y, \dots) - \lambda(g(x, y, \dots) - c)$$
- Set the gradient of $\mathcal{L}$ equal to the zero vector.

  $$\nabla\mathcal{L}(x, y, \dots, \lambda) = \mathbf{0}$$

  In other words, find the critical points of $\mathcal{L}$ (where the derivatives in orthogonal directions are all zero).
- Consider each solution, which will look something like $(x_0, y_0, \dots, \lambda_0)$. Plug each one into $f$. Or rather, first remove the $\lambda_0$ component, then plug it into $f$, since $f$ does not have $\lambda$ as an input. Whichever one gives the greatest (or smallest) value is the maximum (or minimum) point.

**Multiple Constraints**

The method of Lagrange multipliers can be extended to solve problems with multiple constraints using a similar argument. The method of Lagrange seeks points not at which the gradient of $f$ is a multiple of any single constraint's gradient necessarily, but in which it is a linear combination of all the constraints' gradients. Thus, there are M scalars $(\lambda_1, \lambda_2, \ldots, \lambda_M)$ which are the Lagrange multipliers and M constraints $(c_1, c_2, \ldots, c_M)$ such that

$$\nabla f(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{M} \lambda_i \, \nabla g_i(x_1, x_2, \ldots, x_N)$$

where $g_i(x_1, x_2, \ldots, x_N) = c_i, i = 1, 2, \ldots, M$.

In this case, the Lagrangian function becomes

$$\mathcal{L}(x_1, x_2, \ldots, x_N, \lambda_1, \lambda_2, \ldots, \lambda_M) = f(x_1, x_2, \ldots, x_N) - \sum_{i=1}^{M} \lambda_i \, [g_i(x_1, x_2, \ldots, x_N) - c_i]$$

Setting the gradient of $\mathcal{L}$ equal to the zero vector where

$$\nabla \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \nabla f(\boldsymbol{x}) - \sum_{i=1}^{M} \lambda_i \, \nabla[g_i(\boldsymbol{x}) - c_i] = \mathbf{0},$$

and $g_i(\boldsymbol{x}) = c_i, i = (1, 2, \ldots, M)$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_N), \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_M)$.

which amounts to solving $N + M$ equations in $N + M$ unknowns.

The constraint qualification assumption when there are multiple constraints is that the constraint gradients at the relevant critical point are linearly independent.

**Constraints with Inequalities - the Karush-Kuhn-Tucker (KKT) conditions**

The KKT conditions describe what happens when $\boldsymbol{x}$ is the optimal solution to the following constrained optimization problem:

- The gradient of the Lagrangian function is the zero vector.
- All constraints are satisfied.
- The inequality constraints satisfy the complementary slackness condition.

The most important of them is the complementary slackness condition. The optimization problem with equality constraint can be solved using Lagrange multiplier which the gradient of the Lagrangian is zero at the optimal solution. The complementary slackness condition extends this to the case of inequality constraint by saying that at the optimal solution $\boldsymbol{x}$, either the Lagrange multiplier is zero or the corresponding inequality constraint is zero (inactive). That is the reason we need to know whether a constraint is active or not is because of the Karush-Kuhn-Tucker (KKT) conditions.

## 11. Appendix 2 – Sum Rate Maximization

The solution to the sum rate maximization problem [2] given by:

$$maximize \sum_{i=1}^{n} w_i \, log_2 \left(1 + \frac{g_i s_i}{N_i}\right) \; over \; s_i$$

$$such \; that \sum_{i=1}^{n} s_i = P$$

$$where \; s_i \geq 0 \;\; \forall \, i$$

is found using the Lagrange multiplier method for the sum rate maximization function above with constraints including inequalities that satisfy the Karush-Kuhn-Tucker (KKT) conditions.

The Lagrangian for the sum rate maximization problem is:

$$\mathcal{L}(\boldsymbol{s}, \boldsymbol{\lambda}, \lambda_0) = -\sum_{i=1}^{n} w_i \, log_2 \left(1 + \frac{g_i s_i}{N_i}\right) + \lambda_0 \left(\sum_{i=1}^{n} s_i - P\right) - \sum_{i=1}^{n} \lambda_i s_i$$

where $\boldsymbol{s} = \{s_1, \dots, s_n\}$ and $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_n\}$.

The first order optimality condition is found by setting the gradient of the Lagrangian with respect to the power variables $s_i$ equal to zero:

$$\nabla_{s_i} \mathcal{L}(\boldsymbol{s}, \boldsymbol{\lambda}, \lambda_0) = \nabla_{s_i} \left[ = -\sum_{i=1}^{n} w_i \, log_2 \left(1 + \frac{g_i s_i}{N_i}\right) + \lambda_0 \left(\sum_{i=1}^{n} s_i - P\right) - \sum_{i=1}^{n} \lambda_i s_i \right] = 0$$

$$= -\frac{g_i}{g_i s_i + N_i} + \lambda_0 - \lambda_i$$

Solving for $s_i$ results in:

$$s_i = \frac{1}{\lambda_0 - \lambda_i} - \frac{N_i}{g_i}$$

Applying the Karush Kuhn Tucker (KKT) condition of complementary slackness to the slack variables

$\{\lambda_1, \dots, \lambda_n\}$

where $\lambda_i s_i = 0$, then either $s_i = 0$ or $\lambda_i = 0$. Therefore $\lambda_i, i = 1, \dots, n$ vanishes for all $s_i > 0$.

$$s_i = \frac{1}{\lambda_0} - \frac{N_i}{g_i} > 0 \; or \; s_i = 0$$

The simplified solution where $[\cdot]^+$ evaluates to values greater than zero is:

$$s_i = \left[\frac{1}{\lambda_0} - \frac{N_i}{g_i}\right]^+ = \max\left(\frac{1}{\lambda_0} - \frac{N_i}{g_i}, 0\right)$$

that disallows negative power values. Let $\alpha = \frac{1}{\lambda_0}$, so that $s_i = \left[\alpha - \frac{N_i}{g_i}\right]^+$.

It can be shown that $\alpha$ corresponds to a "water-filling" power level whereby all $s_i$ can be found as the difference between this water-filling level and the inverse of the channel gain divided by the channel noise level.

This is an implicit solution as the water-filling level $\alpha$ can be found subject to the maximum power constraint $\sum_{i=1}^{n} s_i = P$ using an iterative procedure such as a bisection search for the value of $\alpha$ that meets this constraint.

# Abbreviations

| | |
|---|---|
| CNR | carrier to noise ratio |
| CWF | conventional water-filling |
| dB | decibel |
| dBmV | decibel millivolt |
| DOCSIS | data over cable service interface specifications |
| EC | echo cancellation |
| FDD | frequency division duplex |
| FDX | full-duplex DOCSIS |
| GWF | geometric water-filling |
| GWFPP | geometric water-filling with sum and individual peak power constraints |
| KKT | Karush-Kuhn-Tucker |
| Mbps | megabits per second |
| MER | modulation error ratio |
| MHz | megahertz |
| OFDM | orthogonal frequency-division multiplexing |
| OFDMA | orthogonal frequency-division multiple access |
| PMA | profile management application |
| QAM | quadrature amplitude modulation |
| SIR | signal to interference ratio |
| SNR | signal to noise ratio |
| SNIR | signal to noise plus interference ratio |
| WFPP | water-filling with sum and individual peak power constraints |

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