THERMAL NOISE REVISITED

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The calculation of the noise figure of a cascade of N unity gain sections is based on the formula NF where F is the amplifier noise figure. A more detailed analysis indicates that this formula is only an approximation. A more exact representation includes the noise figure of the passive elements (coax, taps, etc.), as well as that of the active components. The noise figure of a coax cable is shown to be equal to $1/L$ where L is the attenuation of the passive component. The calculation of the thermal noise for forward and reverse transmission of a cable network is made more precise through the consideration of the noise figure of a matched passive element. In addition to providing a more complete understanding of system noise figure for nonunity gain configurations, the resulting formulation also provides an opportunity for introducing additional parameters in the design of cable networks.

INTRODUCTION

The design of cable systems continuously involves, among other issues, calculations of accumulated noise. The interest in this subject is becoming much greater as larger systems are planned to provide both downstream and upstream transmission of data, audio, and TV for new services. The generation of upstream noise presents a new set of problems for the system designer as compared to the usual treatment of downstream transmission. The objective of this paper is to provide a review of some of the fundamentals of these noise calculations, and to show the advantage of a more unified method of calculating the noise for different configurations of amplifiers, taps and cable. Particular attention is given to the use of a noise factor for the passive as well as the active elements of the network. This method leads to a more exact formula for the noise accumulation of unit gain cascades, and trunk bridger/extender links.

NOISE SOURCE

The noise that appears in a cable network is generated by several different sources. The thermal noise of passive and active components, the ingress of undesired radiation external to the system, the generation of excess noise by the devices connected at each drop, and the intermodulation noise due to nonlinearities in the system are among the most significant components. For the purposes of this article, attention will be directed to the analytic foundation associated with the calculations of the thermal noise build up.

The thermal noise is attributed to the random motion of charge carriers in the resistive elements of the network and to the fluctuations in the currents within the active devices. The noise voltage generated by a resistor at a given temperature can be calculated directly from the value of the resistor. The contribution to the noise from the active devices is, however, generally determined from a measurement of the amplifier noise characteristics (i.e. noise factor or noise figure) rather than a detailed calculation based on a model of the internal noise sources.

The noise generated by a resistor R at temperature T is described by the equivalent circuit shown in Fig. 1. The noise source is considered to be a generator in series with a "noiseless" resistor R. The <u>open circuit</u> RMS voltage e_n is derived from the well-established formula,

$$\overline{e_n^2} = 4KTBR \qquad (1)$$

where
K = Boltzman's constant
T = Absolute temperature
B = Measurement bandwidth

Another parameter defined for the resistor is the maximum available noise power which can be delivered to a load. The maximum power condition occurs when a matched load R is placed across the original resistor. From the simple circuit shown in Fig. 1b the noise power delivered to the matched load is the familiar value.

$$\overline{e_n^2}/4R = KTB \tag{2}$$

Equation (1) can be used for calculating the output noise power in circuits containing multiple resistive components. For example, the matched load resistor in Fig. 1b also serves as a generator of noise. Figure 2 shows the inclusion of a second noise generator whose maximum available power is also KTB. The output noise voltage across terminals a-b is calculated from $\overline{e_{nt}^2} = \overline{e_{o_1}^2} = \overline{e_{o_2}^2}$ where e_{o_1} and e_{o_2} are the noise voltages due to each noise generator. This procedure is valid since the equivalent noise generators are uncorrelated. For the circuit in Fig. 2 the result would be $\overline{e_{n_t}^2} = 2KTBR$.

The important point of this exercise is that the "available noise power" of a device is not necessarily the noise power that would appear across the output terminals when the matched load is actually placed across the device unless the load is assumed to be "noiseless".

The calculation of the maximum available power from a more complex array of passive and active components follows the same procedure. A general two-port device is shown in diagram Fig. 3a. The objective is to define a parameter which would characterize the output noise power that would be delivered to a matched load as shown in Fig. 3b. A known noise source (usually a resistor) is placed at the input terminals. The maximum available output noise power is then calculated. If the input of the network is matched to the resistor, the input noise power is KTB. The total maximum available output noise power is given by

$$N_o = FGKTB, \text{ where} \tag{3}$$

G is the maximum matched power gain or loss of the network, and F is the "Noise Figure" of the network which is related to the excess noise added by the network. The output power, N_o, consists of the contribution due to the external resistor, GKTB, plus the network noise N_m, i.e.,

$$N_o = GKTB + N_m = FGKTB \tag{4}$$

or

$$N_m = (F-1)\ GKTB \tag{5}$$

If two matched networks are cascaded as shown in Fig., 4, one can use the same approach to obtain the usual formula for the effective noise figure, F_e, for the cascade which is defined in the following equation:

$$N_o = F_e G_1 G_2 KTB \tag{6}$$

where N_o is the total maximum available noise power at the output of the network if the input noise is due to an external resistor. At the junction of the two matched networks, the noise power input to the second network is $F_1 G_1 G_2 KTB$. From (5) the contribution of the second network to the output noise is:

$$(F_2-1)\ G_2 KTB \tag{7}$$

Therefore,

$$N_o = F_1 G_1 F_2 KTB + (F_2-1)\ G_2 KTB \tag{8}$$

$$\text{or } F_e = F_1 + \frac{F_2 - 1}{G_1} \tag{9}$$

For a cascade of n stages, one obtains the familiar formula

$$F_e = F_1 + \frac{F_2-1}{G_1} + \frac{F_3-1}{G_1 G_2} + \ldots + \frac{F_n-1}{G_1 \ldots G_{n-1}} \tag{10}$$

Equation (10) is normally discussed in detail for networks containing active components each with their respective noise figures and available gains.

MATCHED PAD NOISE FACTOR

The concept of a noise figure is, however, equally valid for matched passive networks. Fig. 5a shows a symmetrically matched pad which introduces a fixed loss L, i.e., its power gain G = 1/L measured from terminals a - a' to b - b'. To find the noise figure of a matched pad, use the equivalent network as shown in Fig. 5b where a noise generator has been added for each resistor in the network. All of the resistors are assumed to be at the same temperature.

The open circuit noise voltage V_{oc} due to each generator are calculated separately. The maximum available output noise power is then equal to $V_{oc}^2/4R_o$ for each generator. The results are summarized in the following equations:

For generator e_o:

$$V_{oc} = e_o R_3 / R \tag{11a}$$

where $R = R_o + R_1 + R_3$

$$P_o = GKTB \quad \text{where } G = \frac{R_3^2}{R^2} = 1/L$$

For generator e_1

$$V_{oc} = e_1 \, R_3/R$$

$$P_1 = GKTB \, R_1/R_o \qquad (11b)$$

For generator e_2

$$V_{oc} = e_2$$

$$P_2 = KTB \, R_2/R_o \qquad (11c)$$

For generator e_3

$$V_{oc} = e_3 \, (1 - R_3/R)$$

$$P_3 = \frac{KTB \, R_3^2 \, (R_o + R_1)^2}{R^2 R_o} \qquad (11d)$$

The total output noise power is then

$$N_o = P_0+P_1+P_2+P_3 = FGKTB \qquad (11e)$$

where F is the noise figure of the pad Since the pad is matched $R_1 = R_2$, and therefore

$$R_o = (R_o+R_1) \frac{R_2}{R} + R_1 \qquad (11f)$$

If the substitutions are made in (11e), then

$$N_o = KTB, \qquad (12)$$

and therefore FG = 1.

Since $G = {}^1/_L$,

$$F=L \qquad (13)$$

In other words, the noise figure of a matched pad is exactly equal to its loss. The output noise power is just equal to the input noise power, even though the input noise power is attenuated by a factor ${}^1/_L$. The additional output noise is contributed by the noise generators within the pad as shown by the terms P_1, P_2, and P_3 in equation (11e). It is also interesting to observe that if the temperature of the pad was at absolute zero, then $P_1 = P_2 = P_3 = 0$. For this case, $N_o = KTB/L$, and $F = 1$. The output noise power of the pad is thus reduced by a factor ${}^1/_L$ compared to the case in which the pad temperature was equal to that of the input resistor.

COAX CABLE NOISE FACTOR

The result also applies to a coaxial cable. The internal resistors which provide the additional noise in this case are the distributed resistors of the cable rather than the lumped resistors of the pad. The derivation of the noise factor for the coax is perhaps more instructive than the one for the matched pad. Fig. 6 shows a transmission line with a noise generating resistor (R_o) at one end and a noiseless resistor (R_o) at the receiving end. For simplicity the cable loss is attributed only to a distributed series resistance equal to r ohm's per unit length. The calculation of the total noise power in the terminating resistor follows the same procedure as for the pad. The details are given in the appendix. The output noise power due to the resistor R_o is

$$N_1 = KTB/L, \text{ and} \qquad (14)$$

the output noise power due to the cable is

$$N_2 = KTB \, (1-\frac{1}{L}), \text{ where} \qquad (15)$$

${}^1/_L$ is the power loss (G) of the coax.

The total noise power is, therefore,

$$N_o = N_1+N_2 = KTB=FGKTB \qquad (16)$$

The noise factor for the coax is

$$F=L, \text{ since } G={}^1/_L \qquad (17)$$

It should be noted that the relationship, F = L, for the pad and the coax is exact only when the temperature of these components is the same as that of the input resistor as assumed in the above calculations. Equations 11 and 16 can be suitably modified to include any differences in temperature.

DIRECTIONAL COUPLER NOISE FACTOR

The directional coupler is another important passive component whose noise factor must also be considered for the general cascaded network. A schematic of a three-port coupler is shown in Fig. 8. The output noise power consists of the input noise at port one, which is attenuated before reaching port two, and the noise generated by the terminating resistor at port three. The latter, however, is severely attenuated due to the directionality of the coupler. Hence the coupler, in the forward direction, reduces to the matched pad case. Therefore, the noise factor from port one to port one two is $F = L_1$, the coupler loss factor.

NOISE FACTOR OF CASCADED AMPLIFIERS, TAPS, AND CABLES

With the preceding argument, it is now relatively straight-forward to calculate the effective noise figure for diffirent configurations of amplifiers, taps, and cables that are encountered in a cable network. It will also be possible to note where simplifying assumptions have been made in the standard formulas used for cascaded networks.

A) Tandem Amplifier and Coax

The effective noise figure for Fig. 9 from eq. 10 is

$$F_e = F + \frac{L-1}{G} \tag{18}$$

where F = noise figure of matched amplifier

G = available power gain of amplifier

L = loss factor of the cable

A standard cable network design uses L = G, i.e. unity gain sections.

For this case, $F_e = F+1 - 1/_L$ (19)

The important case of a cascade of N unity gain sections can be calculated by substituting eq. 19 into eq. 10. The result is

$$F_e = NF + 1 - \frac{N}{L} \tag{20}$$

This differs from the conventional formula by the added term 1-N/L. However, for practical situations in which $L \sim 100$, $N \sim 20$, and $F \sim 10$, equation 20 reduces to the commonly used formula

$$F_e = NF \tag{21}$$

The derivation of equations 19 and 21 shows the significance of the cable noise factor. As an extreme example, let us evaluate the noise factor of a cascade of unity gain sections when the amplifier is ideal, i.e. F = 1. Eq. 21 yields an effective noise factor of N, i.e. there is a noise buildup even with a cascade of ideal amplifiers connected by lossy cables. The role played by the cable noise is more evident in eq. 25 when F = 1.

A clearer picture of this idealized case can be obtained by tracing the noise through a single unity gain section. The input noise is KTB. The noise at the output of the ideal amplifier is GKTB, which is attenuated by 1/G in the coax, and appears at the output as KTB. If there were no other contribution to the noise, the succeeding cascades would always produce KTB at the output without any buildup. This would be the equivalent to the case where the cable had no noise (i.e., F = 1), but maintained its attenuation L. However, the cable does contribute an additional (1-1/L) KTB to the output noise as shown previously. In the practical case, however, of noisy amplifiers, the noise buildup is due primarily to the amplifier as shown in eq. 21.

B) Tandem coax and directional coupler

A typical configuration is shown in Fig. 10 with two lengths of coax connected by a directional coupler. Eq. 10 yields the effective cascaded noise factor

$$F_e = L_1 + \frac{L_2-1}{1/L_1} + \frac{L_3-1}{1/L_1 1/L_2} \tag{22}$$

$$= L_1 L_2 L_3$$

The network gain is

$$G = \frac{1}{L_1 L_2 L_3} \tag{23}$$

The available output noise power for this combination is

$$N_o = FGKTB = KTB \tag{24}$$

Therefore a cascade of matched taps and coax cable can be considered from a thermal noise viewpoint as the equivalent of a single coax with the same loss factor as the cascade.

C) Trunk with Bridger and Extenders

Another familiar configuration is shown in Fig. 11 in which a main trunk amplifier also contains a bridger amplifier to feed extender amplifiers for local distribution. The transmission path consists of unity gain sections in the trunk and extender sections coupled by a loss L between the trunk amplifier and the bridger. This latter factor needs to be taken into account in deriving the expression for the accumulated noise from the headend to the end of the extender. The system parameters in Fig. 11 are defined as follows:

G_T = trunk amplifier gain

F_T = trunk amplifier noise factor

L = loss factor trunk to bridger

F_B = bridger and extender noise factor

N_T = number of trunk amplifiers in the cascade (excluding the one containing the bridger).

N_B = total number of bridger and extender amplifiers in one extender leg

The noise factor of the trunk cascade from A to B is $N_T F_T + 1$. This is derived from eq. 20 assuming $N/_L$ 1. The noise factor of the trunk to bridger (points B to C in) is $F_T + L-1/G_T$. The noise factor of the extender leg from C to D is $N_B F_B + 1$. The effective noise factor F_e of the three regions in cascade is calculated from eq. (10).

$$\therefore\ Fe = N_T F_T + 1 + F_T + \frac{L-1}{G_T} + \frac{N_B F_B}{G_T \times 1/L} \tag{25}$$

$$\text{or } Fe = (N_T+1)\, F_T + \frac{L}{G_T} N_B F_B + \frac{L-1}{G_T} + 1 \tag{26}$$

The usual formula for Fe considers only the unity gain sections to yield

$$Fe' = (N_T + 1)\, F_T + N_B F_B \tag{27}$$

Eq. 26 approximates Eq. 27 only for the case where L=G, i.e. with a unity gain section between the trunk and bridger. The contribution of the bridger and extender sections to the downstream noise is reduced by the factor L/G. For the case in which L = 10 (10dB) and G = 100 (20 d B), L/G = 1/50. Therefore the effect of the bridger/extender section is even less than that calculated in the standard manner.

D) Branch Network

In the previous illustrations the noise factor was considered for simple paths in which thermal noise from other branches did not enter into the main path. The latter occurs when there is upstream transmission from many points to a single head-end location as shown in Fig. 12. The illustration contains a main trunk with bridger amplifiers driving extender amplifiers with taps distributed along the extender path. The calculation uses the same technique developed for individual sections, i.e. calculate the noise factor and gain for each path, and then calculate the noise at the end of the path due to a resistor at the beginning of the path using No = FGKTB. Where several paths join via a directional coupler, the noise contribution of each path is then added at the common point. A typical path is the last extender leg before the first return amplifier. At the last tap, the terminated drop cable generates a thermal noise power of KTB at the input to the tap which has a tap loss of $1/L_1$. The extender coax also generates KTB at the downstream side of the tap which has a through loss of $1/L_2$. At the upstream leg of the tap the total noise power would be KTB $(\frac{1}{L_1} + \frac{1}{L_2})$. However, under a previous assumption $1/L_1 + 1/L_2 = 1$. Therefore, the next section of extender coax generates KTB as the input to the next tap. Hence no matter how many drops there are on the leg, the input thermal power to the last extender amplifier (A) will be KTB. A tapped trunk section with no active devices on the taps, generates the same amount of noise power in the upstream direction as a cable with the equivalent loss. The noise factor F_1 for the unity gain sections from A to B is

$$F_1 = N_e F_e + 1, \text{ where} \tag{28}$$

N_e = number of extender amplifiers (Excluding the bridger)

F_e = noise factors of extender amplifiers

The noise factor from B to C is

$$F_2 = F_e + \frac{L-1}{G_B}, \text{ where} \tag{29}$$

L = loss factor between bridger output and trunk input

G_B = bridger power gain

The effective noise factor from A to C is therefore

$$F_E = (N_e+1)F_e, \tag{30}$$

neglecting the term $1 + \frac{L-1}{G_B}$.

The return noise power at A due to the extender section is

$$F_E G_e KTB = (N_e+1)F_e \frac{G_B}{L} KTB. \tag{31}$$

Except for a small correction factor, eq. 31 is also the noise power of the extender section at D, since the gain is one from C to D.

The upstream noise power due to a single trunk leg is $N_T F_T$KTB, where N_T is the total number of trunk amplifiers up to D.

The total return noise power is

$$(N_T F_T + n\, G_{B/L}\, (N_e+1)F_e)KTB, \tag{32}$$

where n is the total number of extender legs connected to trunk. The factor $G_{B/L}$ controls the relative contribution of the extender legs to the upstream thermal noise.

CONCLUSION

The calculation of the accumulation of thermal noise in a broadband network is made more precise through the explicit introduction of a noise factor for the passive as well as the active components. When combined with the formula for the noise factor of a cascade of elements, it provides an accurate picture of the approximations that are made in applying the usual CATV thermal noise formulas. The use of a noise factor for a passive element is extremely useful in situations in which the cascade does not consist soley of unity gain sections. The resulting formulations provide an opportunity for introducing additional parameters for cable network designs.

APPENDIX

The noise factor of the coax cable is calculated from the equivalent circuit of a differential length of cable shown in Figs. 6 and 7.

At the transmitting end

$$\overline{e_{n_1}^2} = 4KTBR \qquad (1)$$

The noise voltage e_n appearing at the input terminals to the coax is $e_n = e_{n_1}/2$ (2)

The noise power into the coax is therefore KTB. If the coax has an attenuation coefficient α, then the voltage at the receiving end is

$$e_r = \frac{e_{n_1}}{2} e^{-\alpha \ell}$$

except for a factor due to the time delay. The noise power developed across the terminating resistor is then

$$\overline{e_r^2}/R_o = KTB\, e^{-2\alpha\ell} \qquad (3)$$

$\therefore$ the power gain (loss) is

$$G = e^{-2\alpha\ell} = 1/L \, .$$

The noise power due to the distributed resistor can be determined from the equivalent circuit shown in Fig. 7 where the two sections of the coax have been replaced by their input impedances R_o. A differential length Δx of cable at a distance x from the termination generates a noise voltage e_x where

$$\overline{e_x^2} = 4KTBr \qquad (4)$$

and the voltage e in Fig. 7 is $e_{x/2}$, since $R_o >> r$.

Hence the noise voltage (e_o) across the terminating resistor due to the noise generated at a distance x from the load is

$$e_o = \frac{e_x}{2} e^{-\alpha x} \qquad (5)$$

The noise power generated per unit length in the terminating resistor is

$$e_o^2/R_o = KTB \frac{r}{R_o} e^{-2\alpha x} \qquad (6)$$

The noise power due to the entire coax is

$$P = KTB \frac{r}{R_o} \int_o^{\ell} e^{-2\alpha x} d_x = \frac{KTB\ r}{2\alpha R_o} (1-e^{-2d\ell}) \qquad (7)$$

For low loss cable or at high frequencies it can be shown that the attenuation coefficient (1)

$$\alpha = \frac{r}{2R_o} \qquad (8)$$

Hence the total noise power from equation 16, 20 and 21 is

$$P_o = KTB = FGKTB \qquad (9)$$

Therefore the noise figure for the coax equals L as in the case of the matched pad, since $G = 1/L$.

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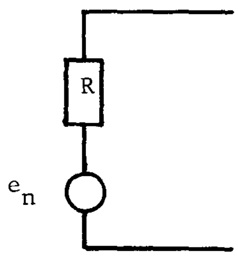

Fig. 1a

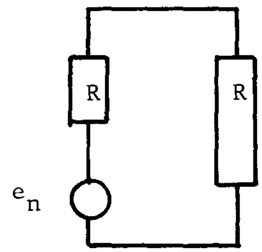

Fig. 1b

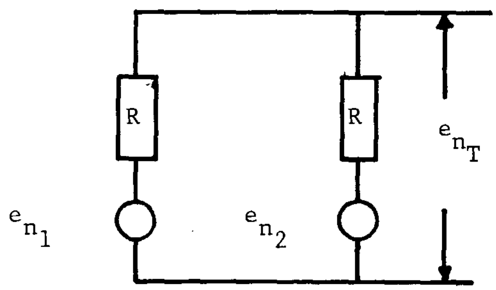

Fig. 2

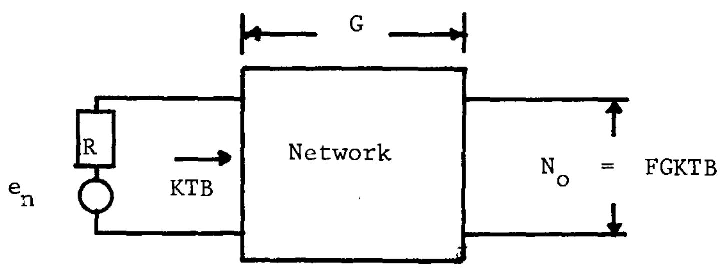

Fig. 3

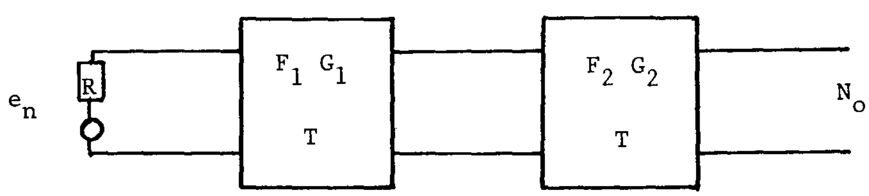

Fig. 4

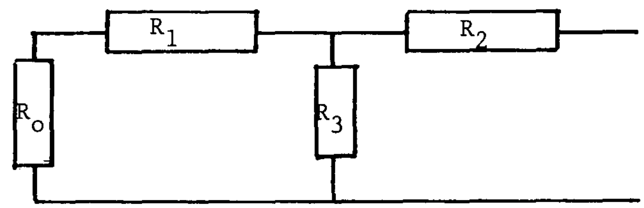

Fig. 5a

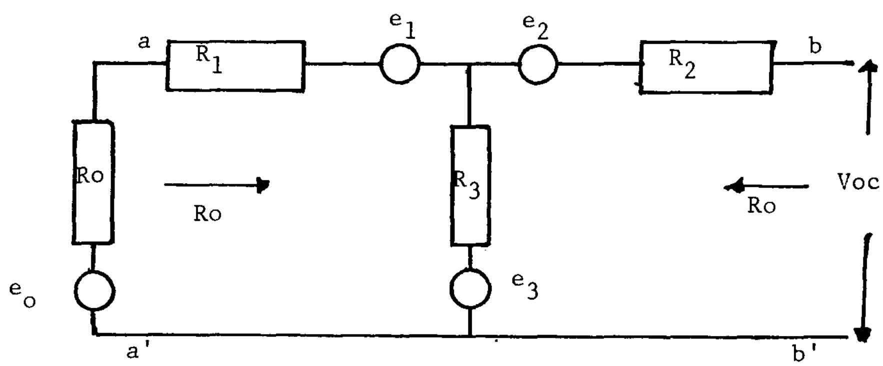

Fig. 5b

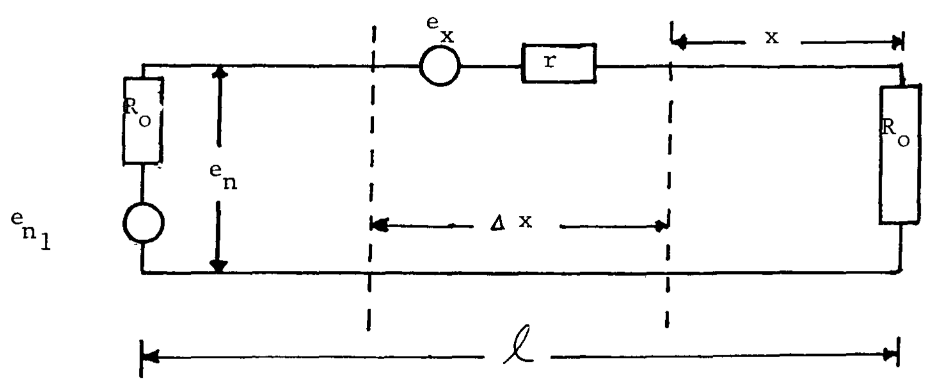

Fig. 6

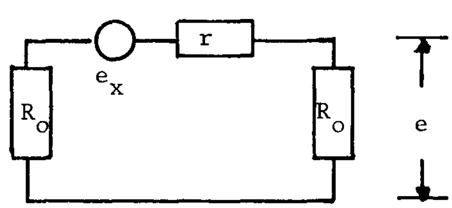

Fig. 7

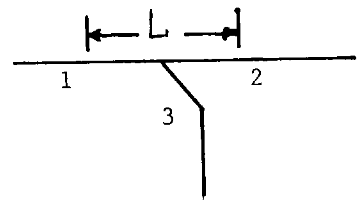

Fig. 8

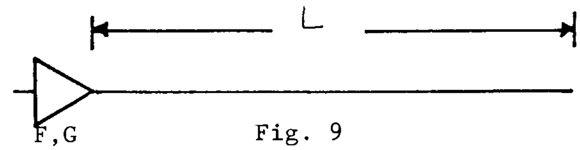

Fig. 9

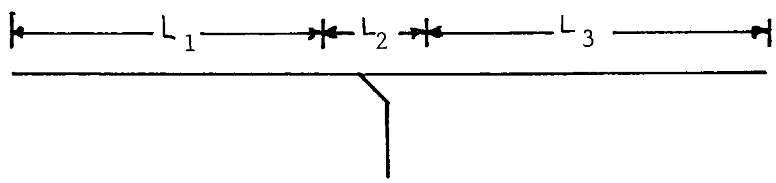

Fig. 10

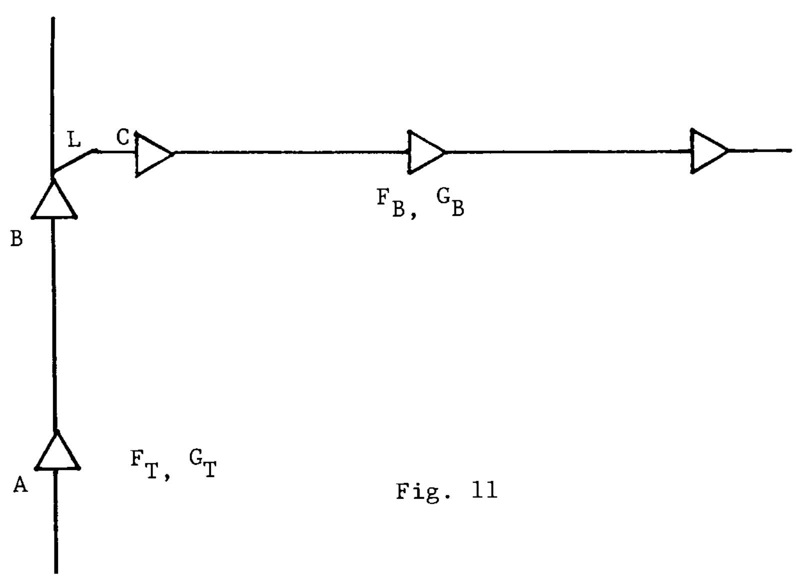

Fig. 11

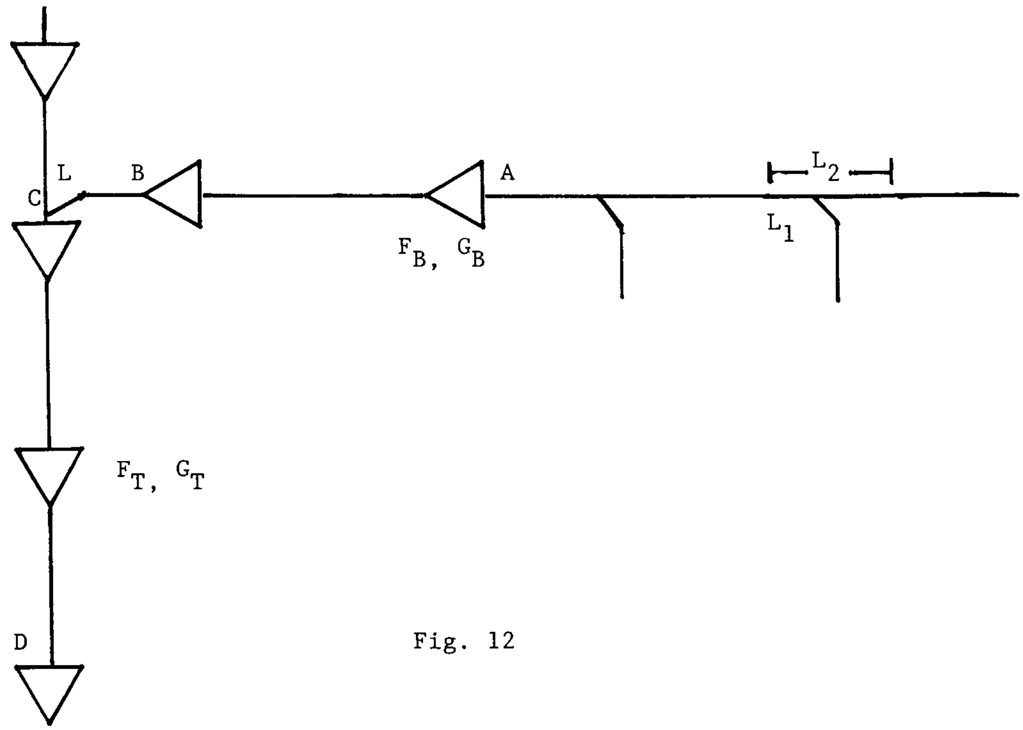

Fig. 12